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Existence Results For Caputo Fractional Integro Differential Equations

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ABSTRACT: In this paper we establish the local and global existence results for the solutions of nonlinear fractional Integro differential equations, using Schauder fixed point theorem and Tychonoff fixed point theorem.

Keywords: Tychonoff fixed point theorem, Schauder fixed point theorem , .

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1 INTRODUCTION

Fractional calculus generalizes traditional calculus to non-integer differential and integral orders. Fractional differential equations provide an excellent tool for description of memory of hereditary properties of various materials and processes.

In recent years, fractional differential equations attracted the attention of many researchers and much work has been done in the past few decades (Podlubny, 1999; Kilbas, A.A. & Srivatsava, H.M. & Trujillo, J. J., 2006; Lakshmikantham, V. & S;Leela & Vasundhara Devi, J., 2009). It is observed that the theory of fractional differential equations model physical phenomena having hereditary properties and its applications can be seen in the mathematical modeling of various physical phenomena, such as heat conduction in materials with memory, viscoelastic materials, quantum evolution of complex systems, (Sabatier J & Agrawal OP & Tenreiro Machado JA, 2007; Sabatier, J, & Melchior, P & Oustaloup, A, 2008) as well as in many fields of science and engineering including fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory and probability. On the other hand Integro differential equations (V.Lakshmikantham & M.Rama Mohan Rao, 1995) also appear to be suitable models to investigate problems in biology and social sciences where memory comes in naturally.

Owing to the fact that both the fore mentioned areas are math models for physical phenomena involving memory or hereditary properties, it is interesting to combine these two areas and investigate the new system which is Fractional Integro differential equations. There has been few papers in this area (Li Huanga & Xian-Fang Li b & Yulin Zhaoa & Xiang-Yang Duana, 2011; Xiaohua Ma & Chengming Huang, 2013) but there has been no systematic investigation of these equations.

In this paper we consider the Caputo fractional integro differential equation of the type

$$ {}^{c}D^{q}x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \, f(s, x(s)) \, ds, \tag{1} $$

$$ x(t_0) = x_0. \tag{2} $$

where $f \in C[I \times \mathbb{R}^n, \ \mathbb{R}^n], \ 0 < q < 1, x \in C^q[[t_0, T], \ \mathbb{R}^n]$, and study its existence results using Schauder fixed point theorem and Tychnoff fixed point theorem.

2 PRELIMINARIES

The basic definitions and theorems that are needed to prove our main result are presented in this section. We begin with the definitions of C_p- Continuity, $R-L$ fractional derivative and Caputo fractional derivative.

2.1 Definition : C_p Continuous

m is said to be C_p continuous if $m \in C_p[[t_0, T], \mathbb{R}]$ that is $m \in C[(t_0, T], \mathbb{R}]$ and $(t-t_0)^p \ m(t) \in C[[t_0, T], \mathbb{R}]$ with $p + q = 1$.

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2.2 Definition: Riemann-Liouville Derivative

For $m \in C_p[[t_0, T], \mathbb{R}]$, the Riemann-Liouville derivative of $m(t)$ is defined as

$$D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{p-1}\, m(s) ds. \tag{3}$$

2.3 Definition: C^q continuous

u is said to be C^q continuous that is $u \in C^q[[t_0, T], \mathbb{R}]$ iff the Caputo derivative of u denoted by ${}^cD^q u$ exists and satisfies

$${}^cD^q u(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds \tag{4}$$

where $0 < q < 1$.

Observe that a function is Caputo differentiable implies that is differentiable. We note that the Caputo and Riemann-Liouville derivatives are related as follows:

$${}^cD^q x(t) = D^q[x(t) - x(t_0)]. \tag{5}$$

For the sake of completeness, we state below some known definitions and theorems related to fixed point theorem.

A completely continuous operator is defined as follows.

2.4 Definition: Completely Continuous Operator

Let X and Y be Banach spaces. A bounded linear operator $T : X \to Y$ is called completely continuous if, an operator T transforms weakly-convergent sequences in X to norm-convergent sequences in Y ..

2.5 Theorem: Schauder Fixed Point Theorem

If E is a closed, bounded, convex subset of a Banach space B and $T : E \to E$ is completely continuous then T has a fixed point.

The next two definitions are useful to apply the Tychonoff Fixed Point Theorem.

2.6 Definition: Locally Convex Vector Space

A locally convex vector space is a pair (X, T) consisting of a vector space X and linear topology T on X which is locally convex, in the sense that every $x \in X$ possesses a fundamental system of convex neighborhoods.

2.7 Definition: Fundamental System of Neighborhoods

A collection γ of neighborhoods of x is called fundamental system of neighborhoods of x if for any neighborhood M of x there exists a finite sequence $V_1,\ V_2,\ \ldots,\ V_n$ of neighborhoods in γ such that $x \in V_i \subseteq M$.

Next, we state the Tychonoff Fixed Point Theorem

2.8 Theorem: Tychonoff Fixed Point Theorem

Let B be a complete, locally convex, linear space and B_0 be a closed convex subset of B. Let the mapping $T : B \to B$ be continuous and $T(B_0) \subset B_0$. If the closure of $T(B_0)$ is compact then T has a fixed point.

3 EXISTENCE RESULTS

In this section, we state and prove local and global existence results for the the IVP for the Caputo fractional integro differential equation given by

$${}^cD^q x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}\, f(s, x(s))\, ds, \tag{6}$$

$$x(t_0) = x_0. \tag{7}$$

where $f \in C[I \times \mathbb{R}^n,\ \mathbb{R}^n],\ 0 < q < 1$; If $x \in C^q[[t_0, T],\ \mathbb{R}^n]$.

We first state and prove the following local existence result by applying Schauder's fixed point theorem.

3.1 Theorem

Assume that $f \in C[J \times \mathbb{R}^n,\ \mathbb{R}^n]$, $\|f(t,x)\| \leq M$ on $D = \{(t,x) : t \in J \ and\ \|x(t) - x_0(t)\| \leq b\}$, where $J = [t_0,\ t_0 + a)$, $x \in \Omega = \{\phi \in C^1[J, \mathbb{R}^n] \ :\ \phi(t_0) = x_0 \ and\ \|\phi(t) - x_0\| \leq b\}$. Then the IVP (6) and (7) possess at least on solution $x(t)$ on $t_0 \leq t < t_0 + \alpha$, where $\alpha = min\{a,\ \{(\frac{b}{M}\Gamma(q+1))^{\frac{1}{2q}}\}\}$

Proof: Consider $D = \{(t,x) : t \in J \ and\ \|x(t) - x_0(t)\| \leq b\}$ and let $\|f(t,x)\| \leq M$ on D. Let $J_0 = [t_0,\ t_0 + \alpha]$, and $\Omega_0 = \{\phi \in C^1[J_0, \mathbb{R}^n] \ :\ \phi(t_0) = x_0 \ and\ \|\phi(t) - x_0\| \leq b\}$, where $|\phi| = \max_{t_0 \leq t \leq t_0+\alpha} |\phi(t)|$. Then Ω_0 is closed, convex and bounded. For any $\phi \in \Omega_0$ define the function $T\phi$ by

$$T\phi(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} \{(t-\tau)^{q-1} [\int_{t_0}^{\tau} (\tau - s)^{q-1}\ \ f(s, \phi(s))\, ds]\}\, d\tau. \tag{8}$$

Clearly $T\phi(t_0) = x_0$.

We shall apply Schauder's fixed point theorem to prove the existence of a fixed point of T in Ω_0, which is equivalent to obtaining the local existence of the solution of the IVP (6) and (7).

For that, first we show that the operator T satisfies the hypothesis of Schauder fixed point theorem. For $t \in J_0$ consider,

$$\begin{aligned}
\|T\phi(t) - x_0\| &= \|\frac{1}{\Gamma(q)} \int_{t_0}^{t} \{(t-\tau)^{q-1} [\int_{t_0}^{\tau} (\tau - s)^{q-1}\ \ f(s, \phi(s))\, ds]\}\, d\tau\| \\
&\leq \frac{M}{\Gamma(q)} \int_{t_0}^{t} \{(t-\tau)^{q-1} [\int_{t_0}^{\tau} (\tau - s)^{q-1}\ \ \ ds]\}\, d\tau \\
&= \frac{M}{q\Gamma(q)} \int_{t_0}^{t} \{(t-\tau)^{q-1}\ (\tau - t_0)^{q}\, d\tau \\
&\leq \frac{M}{\Gamma(q+1)} \int_{t_0}^{t} \alpha^{q-1}\ \alpha^{q}\, d\tau \\
&= \frac{M}{\Gamma(q+1)} \alpha^{2q-1}\ (t - t_0) \\
&\leq \frac{M}{\Gamma(q+1)} \alpha^{2q} \\
&\leq b.
\end{aligned}$$

Therefore $T\phi \in \Omega_0$, which implies that $T\Omega_0 \subseteq \Omega_0$. Next to show equicontinuity of the family $\{T\phi\}$, for given $\epsilon > 0$ choose $\delta = \frac{\epsilon^{\frac{1}{q}}}{\alpha}(\frac{q\Gamma(q+1)}{2M})$. Then, for $t_1,\ t_2 \in J_0$ such that $t_2 > t_1$ consider,

$$\begin{aligned}
&\|T\phi(t_2) - T\phi(t_1)\| \\
&= \|\frac{1}{\Gamma(q)} \int_{t_0}^{t_2} \{(t_2-\tau)^{q-1} [\int_{t_0}^{\tau} (\tau - s)^{q-1}\ \ f(s, \phi(s))\, ds]\}\, d\tau \\
&\qquad -\ \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} \{(t_1-\tau)^{q-1} [\int_{t_0}^{\tau} (\tau - s)^{q-1}\ \ f(s, \phi(s))\, ds]\}\, d\tau\| \\
&= \|\frac{1}{\Gamma(q)}||\ \| \int_{t_0}^{t_1} \{[(t_2-\tau)^{q-1} - (t_1-\tau)^{q-1}][\int_{t_0}^{\tau} (\tau - s)^{q-1}\ \ f(s, \phi(s))\, ds]\}\, d\tau \\
&\qquad + \int_{t_1}^{t_2} (t_2-\tau)^{q-1} [\int_{t_0}^{\tau} (\tau - s)^{q-1}\ \ f(s, \phi(s))\, ds]\, d\tau\| \\
&\leq \|\frac{M}{\Gamma(q)}||\ \int_{t_0}^{t_1} \{\|(t_2-\tau)^{q-1} - (t_1-\tau)^{q-1}\| \int_{t_0}^{\tau} \|(\tau - s)^{q-1}\| ds\}\, d\tau \\
&\qquad + \int_{t_1}^{t_2} \|(t_2-\tau)^{q-1}\| [\int_{t_0}^{\tau} \|(\tau - s)^{q-1}\|\, ds]\, d\tau] \\
&\leq \|\frac{M}{q\Gamma(q)}||\ \int_{t_0}^{t_1} \{\|(t_2-\tau)^{q-1} - (t_1-\tau)^{q-1}\| \|(\tau - s)^{q}\|\}\, d\tau
\end{aligned}$$

$$+\int_{t_1}^{t_2} \|(t_2-\tau)^{q-1}\|(\tau-s)^q\|\, d\tau$$

$$\leq \|\frac{M\,\alpha^q}{\Gamma(q+1)}\| \int_{t_0}^{t_1}\{\|(t_2-\tau)^{q-1}-(t_1-\tau)^{q-1}\|\}\, d\tau + \int_{t_1}^{t_2} \|(t_2-\tau)^{q-1}\|\, d\tau$$

$$= \frac{M\,\alpha^q}{\Gamma(q+1)}[(t_1-t_0)^q+(t_2-t_1)^q-(t_2-t_0)^q+(t_2-t_1)^q]$$

$$\leq \frac{M\,\alpha^q}{q\Gamma(q+1)}[2(t_2-t_1)^q]$$

$$< \epsilon$$

when ever $\|t_2-t_1\| < \delta = \frac{\epsilon^{\frac{1}{q}}}{\alpha}(\frac{q\Gamma(q+1)}{2M})$

i.e., $T(\Omega_0)$ is an equicontinuous family and thus we conclude that the closure of $T(\Omega_0)$ is compact.

To show that T is a continuous map, let us take an $\epsilon > 0$ and $\phi,\ \psi$ be two continuous functions in Ω_0, $t \in J_0$,

$\|T\phi(t) - T\psi(t)\| = \|\frac{1}{\Gamma(q)}\int_{t_0}^{t}\{(t-\tau)^{q-1}[\int_{t_0}^{\tau}(\tau-s)^{q-1}\ [f(s,\phi(s))\ -\ f(s,\psi(s))]\, ds]\}\, d\tau\|$.

Since f is uniformly continuous for the above $\epsilon > 0$, there exists a $\delta_1 > 0$ such that when ever $\|\phi(t)-\psi(t)\| < \delta_1$, we have

$$\|f(s,\phi(s))\ -\ f(s,\psi(s))]\ \| < \frac{\epsilon\Gamma(q+1)}{\alpha^{2q}}.$$

Therefore

$$\|T\phi(t) - T\psi(t)\| < \tfrac{\epsilon\Gamma(q+1)}{\alpha^{2q}}\tfrac{1}{\Gamma(q)}\int_{t_0}^{t}\{(t-\tau)^{q-1}[\int_{t_0}^{\tau}(\tau-s)^{q-1}\, ds\ \}\, d\tau.$$

$$< \tfrac{\epsilon}{\alpha^{2q}}\int_{t_0}^{t}(t-\tau)^{q-1}(\tau-t_0)^q\ \ d\tau.$$

$$\leq \tfrac{\epsilon}{\alpha^{2q}}(\alpha)^{2q-1}(t-t_0)$$

$$\leq \epsilon,$$

when ever $\|\phi(t)-\psi(t)\| < \delta_1$.

Thus the operator T satisfies the hypothesis of Schauder fixed point theorem and hence there is a fixed point of T in Ω_0 which is a solution of IVP (6) and (7).

We shall next discuss a global existence result for IVP (6) and (7) using Tychonoff's fixed point theorem, which we state in the following form.

3.2 Theorem

Assume that
(H_1) $f \in C[\mathbb{R}_+ \times \mathbb{R}^n,\ \mathbb{R}^n]$, $g \in C[\mathbb{R}_+^2,\ \mathbb{R}_+]$, $g(t,u)$ is monotone nondecreasing in u for each $t \in J = [t_0,\ t_0+a]$ and

$$\|f(t,x)\| \leq g(t,\|x\|),$$

(H_2) for every $u_0 > 0$, the scalar Caputo fractional integro differential equation

$$^cD^q u(t) = \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\ g(s,u(s))\, ds, \tag{9}$$

$$u(t_0) = u_0 \tag{10}$$

has a solution $u(t)$ existing for $t \geq t_0$. Then for every $x_0 \in \mathbb{R}^n$ such that $\|x_0\| \leq u_0$, there exists a solution $x(t)$ of the IVP (6) and (7) satisfying $\|x(t)\| \leq u(t)$ for $t \geq t_0$.

Proof: Let us consider the real vector space B of all continuously differentiable functions from $[t_0, \infty)$. Let T be the topology on B induced by the family of pseudo norms $\{V_n(x)\}_{n=1}^{\infty}$ where for $x \in B$,

$$V_n(x) = sup_{t_0 \le t \le n} \|x(t)\|.$$

For this topology $\{S_n\}_{n=1}^{\infty}$ where $S_n = \{x \in B/V_n(x) \le 1\}$ is fundamental system of neighborhoods. Under this topology B is a complete, locally convex linear space. Now define

$$B_0 = \{x \in B : \|x(t)\| \le u(t),\ t \ge t_0\}$$

where $u(t)$ is a solution of scalar Integro Caputo fractional differential equation (6) and (7) existing for $t \ge t_0$. Clearly $B_0 \subseteq B$. Also B_0 is closed, convex and bounded in the topology of B. Now define an integral operator T on B by

$$T\phi(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} \{(t-\tau)^{q-1}[\int_{t_0}^{\tau} (\tau - s)^{q-1} \ \ f(s, \phi(s))\, ds]\}\, d\tau. \tag{11}$$

Then a fixed point of T corresponds to a solution of the Caputo fractional integro differential equation (6) and (7).

Now we prove that the operator T satisfies the hypothesis of Tychonoff's fixed point theorem. The operator T is defined above is compact in the topology of B and B_0 is a bounded subset of B. Hence the closure of $T(B_0)$ is compact. Now to prove $T(B_0) \subseteq B_0$, consider any $x \in B_0$.

$$\begin{aligned}
\|Tx(t)\| &= \|x_0\| + \|\frac{1}{\Gamma(q)} \int_{t_0}^{t} \{(t-\tau)^{q-1}[\int_{t_0}^{\tau} (\tau - s)^{q-1} \ \ f(s, x(s))\, ds]\}\, d\tau\|, \\
&\le \|x_0\| + \frac{1}{\Gamma(q)} \int_{t_0}^{t} \{(t-\tau)^{q-1}[\int_{t_0}^{\tau} (\tau - s)^{q-1} \ \ \|f(s, x(s))\|)\, ds]\}\, d\tau \\
&\le u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} \{(t-\tau)^{q-1}[\int_{t_0}^{\tau} (\tau - s)^{q-1} \ \ g(s, \|x(s))\|)\, ds]\}\, d\tau \\
&\le u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} \{(t-\tau)^{q-1}[\int_{t_0}^{\tau} (\tau - s)^{q-1} \ \ g(s, \|u(s))\|)\, ds]\}\, d\tau \\
&= u(t).
\end{aligned}$$

Using the monotonicity of g, the definition of B_0 and the fact that $u(t)$ is a solution of scalar Caputo fractional integro differential equation. There fore

$$\|Tx(t)\| \le u(t)$$

and hence

$$T(B_0) \subset B_0.$$

Therefore the operator satisfies the hypothesis of Tychonoff's fixed point theorem, T has a fixed point in B_0. Thus the proof is complete.

REFERENCES

[1] Podlubny, I., *Fractional Differential Equations,* Academic Press, San Diego, 1999.

[2] Kilbas,A. A., Srivatsava, H. M. and Trujillo,J. J., *Theory and Applications of Fractional Differential Equations,* Elsevier, Amsterdam, 2006.

[3] Lakshmikantham, V., S;Leela and Vasundhara Devi, J., *Theory of Fractional Dynamic Systems,* Cambridge Scientific Publishers, Colterham, 2009.

[4] Sabatier J, Agrawal OP, Tenreiro Machado JA (eds.): *Advances in Fractional Calculus, Theoretical Developments and Applications in Physics and Engineering,* p. xiv+552. Springer, Dordrecht, The Netherlands (2007).

[5] Sabatier, J, Melchior, P, Oustaloup, A: *A testing bench for fractional order systems education.* Journal Europeen des Systemes Automatises. 42(68), 839861 (2008).

[6] V.Lakshmikantham, M.Rama Mohan Rao, Theory of integro differential Equations, Gordon and Breeach Science Publishers, S.A.,1995.

[7] Li Huanga, Xian-Fang Li b, Yulin Zhaoa, Xiang-Yang Duana, Approximate solution of fractional integro-differential equations by Taylor expan sion method, Computers and Mathematics with Applications, 62 (2011) 1127-1134.

[8] Xiaohua Ma, Chengming Huang, Numerical solution of fractional integro- differential equations by a hybrid collocation method, Applied Mathematics and Computation, Volume 219, Issue 12, 15 February 2013, Pages 6750-6760.