GJMS Special Issue for Recent Advances in Mathematical Sciences and Applications-13 GJMS Vol 2. 2 71–75

Generalized Quasilinearization for Graph Differential Equations through its Associated Matrix Differential Equations

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ABSTRACT: Networks are one of the basic structure in many physical phenomena pertaining to engineering applications. As a network can be represented by graph which is isomorphic to its adjacency matrix, the study of analysis of networks involving rate of change with respect to time reduces to the study of graph differential equations and its associated matrix differential equations. In this paper we develop the method of generalized quasilinearization for graph differential equations through its associated matrix differential equations.

Keywords: Pseudo simple graph, Graph differential equation, Matrix differential equation, Quasilinearization technique.

2013 Subject Classification: 05C20, 34G20.

1 INTRODUCTION

Any natural or man-made system involves interconnections between its constituents, thus forming a network, which can be expressed by a graph (D.D.Siljak & J.V. Devi et al., 2008 & 2013). Graphs have been utilized to model organizational structures in social sciences. It has been noted that the graphs which are static in nature is not suitable for social phenomena whose changes with time are natural. Thus led to the introduction of a dynamic graph and a graph differential equation(GDE). The introduced concepts were successfully studied stability of complex dynamic systems through its associated adjacency matrix.

In (J.V. Devi et al., 2013) the authors proved that the set of all weighted directed simple graphs $(D_N, +, .)$ is a linear space and the set of all corresponding adjacency matrices $(E_N, +, .)$ is a matrix linear space where '+' denotes matrix addition and '.' denotes scalar multiplication using the concepts defined in (D.D.Siljak, 2008). Also they consider a weighted directed simple graph as basic element and developed the theory. Further, they obtained existence and uniqueness solutions of a GDE through its associated MDE using the monotone iterative technique. Further, they developed significant results, the basic concept involved was weighted directed simple graph. Since a simple graph has no loops, this fact when translated to differential equations frame work there is no way to accommodate the rate of change in the weight of an edge e_{ij} for all i, j and its relation with the weights of other edges including the edge e_{ij} for all i, j. This drawback was handled in (J.V. Devi & R.V.G Ravi Kumar, 2014).

In (J.V. Devi & R.V.G Ravi Kumar, 2014) the authors introduced the concepts of pseudo simple graph and product of two graphs and further, since there exists an isomorphism between graphs and their adjacency matrices, they successfully exploited it. A good example , will go along way in support of the theory, they considered the prey predator problem and developed the corresponding matrix differential equation and showed how the nonlinearity is preserved in this set up.

In this paper we proposed to obtain a unique solution of the graph differential equation in few steps. In the first step we converting the GDE to MDE. Next step we develop the generalized quasilinearization for MDE, thus obtaining a unique solution for the MDE. In the final step we consider the matrix function which is a solution of the MDE and construct the corresponding graph function which will be a solution of the GDE.

2 PRELIMINARIES

In this section, we give certain definitions,notations,results and preliminary facts related to GDEs that are required in later.

2.1 Definition : Pseudo simple graph

A simple graph having loops is called as a pseudo simple graph.

Analogous to theory of directed simple graphs developed in (J.V. Devi et al., 2013) we proceed to develop the results in this set up. We avoid the details for fear of repetition.

Let $v_1, v_2, ...v_N$ be N verticess,where N is any positive integer. Let D_N be the set of all weighted directed pseudo simple graphs D=(V, E). Then $(D_N, +, .)$ is a linear space w.r.t the operations + and . defined in (J.V. Devi et al., 2013).

Let the set of all corresponding adjacency matrices be E_N. Then $(E_N, +, .)$ is a matrix linear space where '+' denotes matrix addition and '.' denotes scalar multiplication. With this basic structure defined, the comparison theorems, existence and uniqueness results of a solution of a MDE and the corresponding GDE follow as in (J.V. Devi et al., 2013).

Received March 5, 2014 IFNA Publishers, Inc.

2.2 Definition: Continuous and differentiable matrix function

(1) A matrix function $E : I \to \mathbb{R}^{n\times n}$ defined by $E(t) = (e_{ij}(t))_{N\times N}$ is said to be continuous if and only if each entry $e_{ij}(t)$ is continuous for all $i, j = 1, 2, \ldots, N$ where $e_{ij} : I \to \mathbb{R}$.
(2) A continuous matrix function E(t) is said to be differentiable if and only if each entry $e_{ij}(t)$ is differentiable for all $i, j = 1, 2, \ldots, N$. The derivative of E(t) (if exists) is denoted by E' and is given by $E'(t) = (e'_{ij})_{N\times N}$.

2.3 Definition : Continuous and differentiable graph function

Let $D : I \to D_N$ be a graph function and $E : I \to \mathbb{R}^{n\times n}$ be its associated adjacency matrix function then
(1) D(t) is said to be continuous if and only if E(t) is continuous.
(2) D(t) is said to be differentiable if and only if E(t) is differentiable.

If for any graph D the corresponding adjacency matrix is differentiable then we say that D is differentiable and the derivative of D(if exists) is denote by D'.

Consider the initial value problem

$$D' = \mathcal{G}(t, D), \quad D(t_0) = D_0 \tag{1}$$

Let E, E_0 be adjacency matrices corresponding to any graph D and the initial graph D_0.

Then the MDE is given by

$$E' = F(t, E), \quad E(t_0) = E_0 \tag{2}$$

where F(t,E) is the adjacency matrix function corresponding to $\mathcal{G}(t, D)$.

2.4 Definition : Solution of a Matrix Differential Equation

Any continuous differentiable matrix function E(t) is said to be a solution of (2), if and only if it satisfies (2).

2.5 Definition: Solution of a Graph Differential Equation

By a solution of GDE (1) we mean the graph function D(t) corresponding to the matrix function E(t) of the MDE (2).

In order to obtain a unique solution of (1) we use the corresponding adjacency MDE. As there exists an isomorphism between graphs and matrices, the solution obtained for the MDE will be a solution of the corresponding GDE.

2.6 Definition: Convergence of a Matrix

Let $\{E_n\}$ be a sequence of matrices and E be a matrix then $\mathrm{E_n}$ converges to E if and only if given $\epsilon > 0$ there exist $\mathrm{n \geq N}$ such that $||\mathrm{E_n - E}|| \leq \epsilon \; for\ all \quad \mathrm{n \geq N}$. This means $\mathrm{e_{n_{ij}}} \to e_{ij}$ for all $1 \leq i, j \leq N$

2.7 Definition

Consider two matrices A and B of order N. We say that $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i, j = 1, 2, \ldots, N$.

With the necessary preliminaries in place, we proceed to the next section to develop the main results.

3 Generalized Quasilinearization for Matrix Differential Equations

In this section we shall construct a monotone sequence that converges quadratic to the solution of

$$\mathrm{E' = F(t, E), \quad E(t_0) = E_0} \tag{3}$$

where $\mathrm{F \in C[I \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}]}$.

First we begin with the definition of lower and upper solutions for (3)

3.1 Definition:

Let $V_0, W_0 \in C^1[I, \mathbb{R}^{n\times n}]$. Then
(i) V is said to be lower solution of a matrix differential equation if and only if

$$\mathrm{V_0' \leq F(t, V_0), \quad V_0(t_0) \leq E_0} \tag{4}$$

and
(ii) W is said to be upper solution of a matrix differential equation if and only if

$$\mathrm{W_0' \geq F(t, W_0), \quad W_0(t_0) \geq E_0} \tag{5}$$

respectively.

3.2 Definition:

Let $F \in C[I \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}]$ such that $F(t,X) = [f_{ij}(t,X)]$, where $f_{ij} : I \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is differentiable w.r.t X_{ij}. Then the partial derivative of $F(t,Y)$ w.r.t X is denoted by $F_X(t,Y)$ and is defined as follows.

$$F_X(t,Y) = \begin{bmatrix} \sum_{i,j} f_{11x_{i,j}}(t,Y) & \dots & \sum_{i,j} f_{1nx_{i,j}}(t,Y) \\ \sum_{i,j} f_{21x_{i,j}}(t,Y) & \dots & \sum_{i,j} f_{2nx_{i,j}}(t,Y) \\ . & & \\ . & & \\ . & & \\ \sum_{i,j} f_{n1x_{i,j}}(t,Y) & \dots & \sum_{i,j} f_{nnx_{i,j}}(t,Y) \end{bmatrix}$$

We first state a couple of Lemmas that are necessary in the proof of our main theorem.

3.3 Lemma

Let $P \in C^1[I, \mathbb{R}^{n\times n}]$ such that $P' \le MP$ and $P(t_0) \le 0$ where $M \in \mathbb{R}^{n\times n}$. Then $P(t) \le 0$

Proof. Consider the linear matrix differential equation

$$P'(t) \le MP, \quad P(t_0) \le 0$$

whose unique solution is given by

$$P(t) \le e^{M(t-t_0)}P(t_0)$$

Then by hypothesis, we get, $P(t) \le 0$

3.4 Lemma

Let

(i) $V_0(t)$ and $W_0(t)$ be lower and upper solutions of the matrix differential equation (3) and

(ii) $V_1(t)$ and $W_1(t)$ respectively be the unique solutions of the linear non-homogeneous matrix differential equations

$$V_1' = F(t,V_0) + F_X(t,V_0)(V_1 - V_0), \quad V_1(t_0) = E_0 \tag{6}$$

$$\text{and } W_1' = F(t,W_0) + F_X(t,V_0)(W_1 - W_0), \quad W_1(t_0) = E_0 \tag{7}$$

Then $V_0(t) \le V_1(t) \le W_1(t) \le W_0(t)$ on I

Proof :

Suppose that $V_0(t)$ is a lower solution of (3) and $V(t)$ be the unique solution of (6).
Set $P = V_0 - V_1,\ t \in [t_0, T]$. Then
$P' \le F(t,V_0) - [F(t,V_0) + F_X(t,V_0)(V_1 - V_0)]$
$\le F_X(t,V_0)P \le MP$
and $P(t_0) \le V_0(t_0) - V_1(t_0) \le 0$

Using Lemma 3.3 we get $P(t) \le 0$. Thus $V_0 \le V_1$ on I.

Now we show that $W(t) \le W_0(t),\ t \in [t_0, T]$.
Suppose that $W_0(t)$ is a upper solution of (3) and $W(t)$ be the unique solution of (7).
Set $P = W_1 - W_0,\ t \in [t_0, T]$. Then
$P' \le F(t,W_0) + F_X(t,V_0)(W_1 - W_0) - F(t,W_0)$
$\le F_X(t,V_0)P \le MP$
and $P(t_0) \le W_1(t_0) - W_0(t_0) \le 0$

By Lemma 3.3, $P(t) \le 0$. Hence $W_1 \le W_0$ on I.

Finally we show that $V_1(t) \le W_1(t),\ t \in [t_0, T]$. Set $P(t) = V_1(t) - W_1(t),\ t \in [t_0, T]$. Then
$P'(t) \le F(t,V_0) + F_X(t,V_0)(V_1 - V_0) - [F(t,W_0) + F_X(t,V_0)(W_1 - W_0)]$
$\le F(t,V_0) - F(t,W_0) + F_X(t,V_0)[V_1 - V_0 - W_1 + W_0]$
$\le F(t,\xi)(V_0 - W_0) + F_X(t,V_0)[V_1 - V_0 - W_1 + W_0]$

$\leq \mathrm{F_X(t,V_0)(V_0-W_0)+F_X(t,V_0)[V_1-V_0-W_1+W_0]}$
$\leq \mathrm{F_X(t,V_0)P \leq MP}$
and $\mathrm{P(t_0)\leq V_1(t_0)-W_1(t_0)\leq 0}$.
Using Lemma 3.3, $P(t)\leq 0$. Thus $\mathrm{V_1(t)\leq W_1(t),\ t\in[t_0,T]}$.
Hence $\mathrm{V_0(t)\leq V_1(t)\leq W_1(t)\leq W_0(t)}$ on I.

3.5 Theorem

Assume that
(i) $\mathrm{V_0,W_0\in C^1[I,\mathbb{R}^{n\times n}]}$ be respectively lower and upper solutions of the IVP for the matrix differential equation (3) such that $\mathrm{V_0(t)\leq W_0(t),\ t\in I}$
(ii) Let $\mathrm{F\in C[I\times\mathbb{R}^{n\times n},\mathbb{R}^{n\times n}]}$ and $\mathrm{F_X(t,X)}$ exists, such that $\mathrm{F(t,X)\geq F(t,Y)+F_X(t,Y)(X-Y)}$, for $\mathrm{X\geq Y}$ and $\mid \mathrm{F_X(t,X)-F_X(t,Y)}\mid\leq \mathrm{L}\mid \mathrm{X-Y}\mid$, $L\in\mathbb{R}^{n\times n}$.
Then there exist monotone sequences $\{\mathrm{V_n}\}$,$\{\mathrm{W_n}\}$ such that $\mathrm{V_n}\to\rho,\mathrm{W_n}\to\mathrm{R}$ as $n\to\infty$ uniformly and monotonically to the unique solution $\rho=\mathrm{R}=\mathrm{X}$ of IVP (3) on $[t_0,T]$ and the convergence is quadratic.

Proof:
Consider the linear matrix differential equation given by,

$$\mathrm{V'_{k+1}=F(t,V_k)+F_X(t,V_k)(V_{k+1}-V_k),\quad V_{k+1}(t_0)=X_0.} \tag{8}$$

and

$$\mathrm{W'_{k+1}=F(t,W_k)+F_X(t,V_k)(W_{k+1}-W_k),\quad W_{k+1}(t_0)=X_0.} \tag{9}$$

Then it follows from Lemma 3.3 that the linear matrix differential equations (8) and (9) have unique solutions V_{k+1} and W_{k+1} respectively, whenever V_k and W_k are known lower and upper solutions of the IVP (3). Further, by setting $\mathrm{k}=0$ in the above system, we apply Lemma-3.4 to obtain that $\mathrm{V_0\leq V_1\leq W_1\leq W_0}$ on $[t_0,T]$. We now claim that

$$\mathrm{V_0\leq V_1\leq ...\leq V_k\leq V_{k+1}\leq W_{k+1}\leq W_k...\leq W_1\leq W_0\ on\ [t_0,T].} \tag{10}$$

Since the result is already proved for $n=0$, we assume that the result holds for $n=k$ and prove it for $n=k+1$, this means that

$$\mathrm{V_{k-1}\leq V_k\leq W_k\leq W_{k-1}} \tag{11}$$

where V_k and W_k are solutions of the IVP's

$$\mathrm{V'_k=F(t,V_{k-1})+F_X(t,V_k)(V_k-V_{k-1}),\qquad V_k(t_0)=X_0.} \tag{12}$$

and

$$\mathrm{W'_k=F(t,W_{k-1})+F_X(t,V_k)(W_k-W_{k-1}),\quad W_k(t_0)=X_0.} \tag{13}$$

Since V_k is a lower solution of (3). Now by using Lemma 3.3, we obtain that V_{k+1} is a unique solution of (8) on $[t_0,T]$ and hence an application of the Lemma 3.4 yields that $V_k\leq \mathrm{V_{k+1}}$ on $[t_0,T]$. Similarly, it can be shown that W_k is an upper solution of (3) and by Lemma 3.3, we obtain that W_{k+1} is a unique solution of (9) on $[t_0,T]$ and hence an application of the Lemma 3.4 gives that $\mathrm{W_{k+1}\leq W_k}$ on $[t_0,T]$. Further, working in the lines of the Lemma 3.4, we obtain that $\mathrm{V_{k+1}\leq W_{k+1}}$ on $[t_0,T]$.

Hence by the principle of mathematical induction, we deduce the relation (10) and our claim holds. Clearly the sequences are uniformly bounded by relation (10), this also yields that the sequences $\{V_n\}$ and $\{W_n\}$ are also uniformly bounded. As a result the sequences $\{\mathrm{V_n}\}$ and $\{\mathrm{W_n}\}$ are equicontinuous on $[t_0,T]$ and therefore by using Ascoli-Arzela Theorem, there exists subsequences $\{\mathrm{V_{n_k}}\}$,$\{\mathrm{W_{n_k}}\}$ that converges uniformly on $[t_0,T]$. In view of (10) it also follows that the entire sequences $\{\mathrm{V_n}\}$,$\{\mathrm{W_n}\}$ converges uniformly to $\rho(\mathrm{t})$ and $\mathrm{R(t)}$ respectively.
Now we can show that ρ and R are solutions of the IVP(3). Since $\mathrm{F_X}$ exists and is bounded on $[t_0,T]$, we obtain that F is Lipschitz and hence the solution is unique. Thus $\rho=\mathrm{X}=\mathrm{R}$ on $[t_0,T]$. Next, our aim is to show that this convergence is quadratic.
Set $\mathrm{P_{n+1}=X-V_{n+1}}$. Then
$\mathrm{P'_{n+1}=F(t,X)-F(t,V_n)-F_X(t,V_n)(V_{n+1}-V_n)}$
$\leq \mathrm{F_X(t,\xi)P_n-[F_X(t,V_n)(-P_{n+1}+P_n)\leq LP_n^2+MP_{n+1}}$ where $\mid \mathrm{F_X(t,V_n)}\mid\leq \mathrm{M}$

$\mathrm{P'_{n+1}\leq L\mid P_n\mid^2+MP_{n+1},\qquad P_{n+1}(0)\leq 0}$

Now using the solution of the linear non homogeneous matrix differential equation, we get

$\mathrm{P_{n+1}(t)\leq e^{M(t-t_0)}\,P_{n+1}(0)+\int_{t_0}^{t}\,Lp_n^2\,e^{M(s-to)}ds}$

$\mathrm{||P_{n+1}(t)||\ \leq L||P_n||^2\int_{t_0}^{t}\ e^{M(s-to)}ds}$

$\leq \mathrm{\frac{L}{M}||P_n||^2\,e^{(t-to)}}$

$$\leq \frac{\mathrm{L}}{\mathrm{M}} ||\mathrm{P_n}||^2 \, \mathrm{e}^{(\mathrm{T-to})}$$

$$\leq \lambda \, ||\mathrm{P_n}||^2 \quad \text{where } \lambda = \frac{\mathrm{L}}{\mathrm{M}} \, \mathrm{e}^{(\mathrm{T-to})}$$

$$||X - V_{n+1}|| \leq \lambda \, ||X - V_n||^2$$

Hence the quadratic convergence of the sequence $\{\mathrm{V_n(t)}\}$ is proved. Similarly, we can prove the quadratic convergence of the sequence $\{\mathrm{W_n(t)}\}$ to the solution $X(t)$ of IVP (3). Hence the proof of our main theorem is completed.

3.6 Theorem

Consider the GDE (1) and its corresponding MDE is (2). Also assume that F(t,E) be the adjacency matrix corresponding to $\mathcal{G}(t, D)$ such that F satisfies $\mathrm{F(t, X)} \geq \mathrm{F(t, Y)} + \mathrm{F_X(t, Y)(X - Y)}, \text{for } \mathrm{X} \geq \mathrm{Y}$ and $| \mathrm{F_X(t, X)} - \mathrm{F_X(t, Y)} | \leq \mathrm{L} \mid \mathrm{X - Y} \mid$, $L \in \mathbb{R}^{n \times n}$. Then their exists a unique solution for the GDE (1).

Proof.
Since F satisfies the hypothesis of Theorem 3.5, the MDE (2) has unique solution say E(t).
Then the corresponding graph function D(t) is the unique solution of the GDE (1).

4 ACKNOWLEDGMENTS

This work was was done under the project no 2/48(8)/2011/- R&D II/1600 sanctioned by National Board of Higher Mathematics, Department of Atomic Energy, Government of India.The author acknowledge their support.

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