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Stability Results for Impulsive Set Differential Equations Involving Causal Operators with Memory

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ABSTRACT: In this paper we study the stability results for impulsive set differential equations involving causal operators with memory by considering initial functions as a Hukuhara difference of two functions. This will enable to obtain results parallel to ordinary differential equations with delay.

Keywords: Impulsive set differential equations, Causal operators, Equi stable, Uniformly stable, Asymptotically stable. .

1 Introduction

Set differential equations are a generalization of ordinary differential equations and vector differential equations in a semilinear metric space and are useful in studying multivalued differential inclusions or multivalued differential equations (Lakshmikantham, V., GnanaBhaskar, T. and Vasundhara Devi, J.,2006). Further set differential equations involving causal operators with memory include the above said special cases for various types of equations such as set integro differential equations, set differential equations with delay so on and such a generalization is interesting as it gives a comprehensive view of different types of differential equations.

Also it is observed in (Gnana Bhaskar, T., Vasundhara Devi, J., 2005) that solutions for set differential equations contain a lot of undesirable information that needs to be seperated, so that the equations in this setup satisfy the stability behaviour similar to that of scalar or vector equations. In order to take care of this situation, the Hukuhara difference in initial values was considered in (Gnana Bhaskar, T., Vasundhara Devi, J., 2005), and stability results were obtained for the initial value problem of set differential equations.

In this paper we extend results in (Vasundhara Devi, J., Appala Naidu, Ch., 2012) to impulsive set differential equations involving causal operators with memory by considering the Hukuhara difference of initial functions. We obtain stability results using Lyapunov-like functions.

2 Preliminaries

We begin with the definition of $K_c\left(\mathbb{R}^n\right)$, the semilinear space in which we work. We next define the Hausdorff metric and Hukuhara difference and proceed to define the Hukuhara derivative and Hukuhara integral. Further we also state some important properties that are useful tools in our paper. We also define a partial order in $K_c\left(\mathbb{R}^n\right)$.

Let $K_c(\mathbb{R}^n)$ denote the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$. Define the Hausdorff metric

$$D[A,B] = \max[\sup_{x\in B} d(x,A), \ \sup_{y\in A} d(y,B)], \tag{1}$$

where $d(x,A) = \inf[d(x,y) : y \in A]$, A and B are bounded sets in $\mathbb{R}^n$. We note that $K_c(\mathbb{R}^n)$ with this metric is a complete metric space.

It is known that if the space $K_c(\mathbb{R}^n)$ is equipped with Minskowskies addition and non-negative scalar multiplication, then $K_c(\mathbb{R}^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space.

The Hausdorff metric (1) satisfies the following properties:

$$D[A+C, B+C] = D[A,B] \text{ and } D[A,B] = D[B,A], \tag{2}$$

$$D[\lambda A, \lambda B] = \lambda D[A,B], \tag{3}$$

$$D[A,B] \leq D[A,C] + D[C,B], \tag{4}$$

for all $A, B, C \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$.

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Let $A, B \in K_c(\mathbb{R}^n)$. The set $C \in K_c(\mathbb{R}^n)$ satisfying $A = B + C$ is known as the Hukuhara difference of the sets A and B and is denoted by the symbol $A - B$. We say that the mapping $F : I \to K_c(\mathbb{R}^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if

$$\lim_{h \to 0+} \frac{F(t_0 + h) - F(t_0)}{h} \text{ and } \lim_{h \to 0+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist in the topology of $K_c(\mathbb{R}^n)$ and are equal to $D_H F(t_0)$. Here I is any interval in $\mathbb{R}$.

With these preliminaries, we consider the set differential equation

$$D_H U = F(t, U), \;\; U(t_0) = U_0 \in K_c(\mathbb{R}^n), \;\; t_0 \geq 0, \tag{5}$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$.

The mapping $U \in C^1[J, K_c(\mathbb{R}^n)]$, $J = [t_0,\ t_0 + a]$ is said to be a solution of (5) on J if it satisfies (5) on J. Since $U(t)$ is continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^{t} D_H U(s) ds, \;\; t \in J. \tag{6}$$

Hence, we can associate with the IVP (5) the Hukuhara integral

$$U(t) = U_0 + \int_{t_0}^{t} F(s, U(s)) ds, \;\; t \in J. \tag{7}$$

where the integral is the Hukuhara integral which is defined as,

$$\int F(s) ds = \{\int f(s) ds : f \text{ is any continuous selector of } F\}$$

Observe also that U(t) is a solution of (5) on J iff it satisfies (7) on J.

We now proceed to define a partial order in the metric space $(K_c(\mathbb{R}^n), D)$. We begin with the definition of a cone in this set up.

Let $K(K^o)$ be the subfamily of $K_c(\mathbb{R}^n)$ consisting of set $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is a non-negative(positive) vector of n components satisfying $u_i \geq 0$ $(u_i > 0)$ for i = 1...n. Then K is a cone in $K_c(\mathbb{R}^n)$ and K^0 is the nonempty interior of K.

2.1 Definition

For any U and $V \in K_c(\mathbb{R}^n)$, if there exists $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K(K^0)$ and $U = V + Z$ then we say that $U \geq V$ $(U > V)$. Similarly we can define $U \leq V$ $(U < V)$.

3 Stability Results

In this section we proceed to study the stability properties of impulsive set differential equations involving causal operators with memory. In order to set up the problem we need the following notation.
Let $\tilde{E} = C[[t_0 - h_1, \infty), K_c(\mathbb{R}^n)]$ with norm

$$\tilde{D}[U, \theta] = \sup_{t_0 - h_1 \leq t < \infty} \frac{D[U(t), \theta]}{h(t)}. \tag{8}$$

Where θ is the zero element in $\mathbb{R}^n$,which is regarded as a point set and $h : [t_0, \infty] \to \mathbb{R}^+$ is a continuous map and further $[\tilde{E}, \tilde{D}]$ is a Banach space.

We define a norm on $\tilde{E}$ as follows: for $U, V \in \tilde{E}$

$$D_0[U, V] = \sup_{t_0 \leq t \leq \infty} D[U(t), V(t)] \tag{9}$$

where D denotes the Hausdorff Metric.

3.1 Definition

By a causal operator or a Volterra operator or a non anticipative operator we mean a mapping Q: $\tilde{E} \to \tilde{E}$ satisfying the property that if $U(s) = V(s)$, $t_0 \leq s \leq t < \infty$ then $(QU)(s) = (QV)(s)$, $t_0 \leq s \leq t < \infty$.
By a causal operator with memory we mean a mapping Q: $\tilde{E} \to \tilde{E}$ such that for U(s) = V(s), $t_0 - h_1 \leq s \leq t < \infty$, then $(QU)(s) = (QV)(s)$, $\quad t_0 - h_1 \leq s \leq t < \infty$.

It is observed in (Lakshmikantham, V., GnanaBhaskar, T. and Vasundhara Devi, J.,2006, Vasundhara Devi, J., Appala Naidu, Ch, 2011) that the Hukuhara difference in the initial function is necessary to obtain results parallel to ordinary differential equations with delay. For this it was assumed that the initial functions given by Φ_0 and Ψ_0 exist such that the Hukuhara difference $\Phi_0 = \Psi_0 + \chi_0$ exists and we can write $\chi_0 = \Phi_0 - \Psi_0$. In order to simplify our study, we consider the set differential equation involving causal operators with memory given by

$$D_H U = (QU)(t), \;\; U(t_0) = \chi_0 \in C_1 \tag{10}$$

We observe from the results developed in (Gnana Bhaskar, T., Vasundhara Devi, J., 2005, Vasundhara Devi, J., Appala Naidu, Ch., 2012) that it is essential to consider the initial function as a Hukuhara difference of two functions so that we obtain results parallel to ordinary set differential equation with delay. Thus we assume that the solutions $U(t) = U(t, t_0, \chi_0)$ is of the form $U(t, t_0, \Phi_0 - \Psi_0)$ where $\Phi_0 = \Psi_0 + \chi_0$.

Consider the nonlinear impulsive set differential equation involving causal operator with memory

$$\begin{cases} D_H U = (QU)(t); & t \neq t_k, \\ U(t_k^+) = I_k(U(t_k)); & t = t_k, \quad k = 1, 2, 3, ... \\ U_{t_0} = \chi_0 \in C_1; \end{cases} \tag{11}$$

where $C_1 = C[[t_0 - h_1, t_0], K_c(\mathbb{R}^n)]$ with metric

$$D_1[\Phi_0, \Psi_0] = \sup_{t_0 - h_1 \leq s \leq t_0} D_1[\Phi_0(s), \Psi_0(s)] \tag{12}$$

In order to describe the system of our study we begin with the following notation.

H_1: $0 < t_1 < t_2 < ... < t_k < ...$ and $t_k \to \infty$ as $k \to \infty$.

H_2: $Q : \tilde{E} \to \tilde{E}$ is continuous in each sub interval $(t_{k-1}, t_k]$ and for each $U \in \tilde{E}$, k=1,2,3,...

$$\lim_{(t,V) \to (t_k^+, U)} (QV)(t) = (QU)(t_k^+)$$

exists.

H_3: $I_k : K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$

Denote $S(\rho) = \{U \in K_c(\mathbb{R}^n) : \tilde{D}[U, \theta] < \rho\}$ and $S^c(\rho), \partial S(\rho)$ are the complement and boundary of $S(\rho)$ respectively. We define the following class of functions.

$K = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a$ is strictly increasing and $a(0) = 0\}$;

$CK = \{a \in C[\mathbb{R}_+^2, \mathbb{R}_+] : a(t, u) \in K \;\; for \;\; each \;\; t \in \mathbb{R}_+\}$;

$V_0 = \{V \in \mathbb{R}_+ \times K_c(\mathbb{R}^n) \to \mathbb{R}_+ : V$ is continuous in $(t_{k-1}, t_k] \times K_c(\mathbb{R}^n)$ and for each $U \in K_c(\mathbb{R}^n), k = 1, 2, 3, ...$

$$\lim_{(t,W) \to (t_k^+, U)} V(t, W) = V(t_k^+, U)$$

exists and V is locally Lipschitz in U$\}$.

For $V \in V_0, \; and \; (t, U) \in (t_{k-1}, t_k] \times K_c(\mathbb{R}^n)$,

define the generalized derivative along the solution of (11).

$$D^+V(t, U) = \lim_{h \to 0^+} \frac{[V(t + h, U + h(QU)(t)) - V(t, U)]}{h} \tag{13}$$

We now proceed to give the definitions of stability in this set up.

3.2 Definition

The trivial solution of the system (11) is said to be equi stable, if for each $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, for any solution $U(t) = U(t, t_0, \chi_0)$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $D_0[\chi_0, \theta] \leq \delta \Rightarrow \tilde{D}[U(t), \theta] < \epsilon, \;\; t \geq t_0$.

3.3 Definition

The trivial solution of the system (11) is said to be uniformly stable, if the δ in definition (3.2) is independent of $t_0 \in \mathbb{R}_+$.

3.4 Definition

The trivial solution of the system (11) is said to be quasi-equi asymptotically stable, if for each $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exist two positive numbers $\delta_0 = \delta_0(t_0, \epsilon) > 0$ and $T = T(t_0, \epsilon)$ such that $D_0[\chi_0, \theta] \leq \delta_0 \Rightarrow \tilde{D}[U(t), \theta] < \epsilon, \;\; t \geq t_0 + T$.

3.5 Definition

The trivial solution of the impulsive differential system (11) is said to be quasi- uniformly asymptotically stable, if T in definition (3.4) is independent of $t_0 \in \mathbb{R}_+$.

Now we prove the following lemma.

3.6 Lemma

Let $V \in V_0$ and suppose that

$$\begin{cases} D^+V(t,U) \leq g(t,V(t,U)),\ t \neq t_k\ and \\ V(t_k^+,U(t_k^+)) \leq \psi_k(V(t_k,U(t_k))),\ \ t = t_k, \end{cases} \tag{14}$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous in each sub interval $(t_{k-1}, t_k]$ so that

$$\lim_{(t,v)\to(t_k^+,u)} g(t,v) = g(t_k^+,u)$$

exists and $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+\ \ is\ non-decreasing.\ Let\ r(t) = r(t,t_0,w_0)$ be the maximal solution of the scalar impulsive differential system

$$\begin{cases} w' = g(t,w), \qquad t \neq t_k \\ w(t_k^+) = \psi_k(w(t_k)),\ \ t = t_k \\ w(t_0^+) = w_0, \quad k = 1,2,... \end{cases} \tag{15}$$

existing on $[t_0,\infty)$. Then $V(t_0^+,\chi_0(t_0)) \leq w_0$ implies that $V(t,U(t)) \leq r(t),\ t \geq t_0$,
where $U(t) = U(t,t_0,\chi_0(t_0))$ is any solution of the system (11) on $[t_0,\infty)$.

Proof. Let $U(t) = U(t,t_0,\chi_0(t_0))$ be any solution of the system (11) for $t \geq t_0$ and consider the interval $(t_k,t_{k+1}]$.
Set $m(t) = V(t,U(t))$ for $t \neq t_0$ so that for small $h > 0$ we have
$D^+m(t) \leq g(t,V(t,U(t)))$ for $t \neq t_k$ provided $V(t_0^+,\chi_0(t_0)) \leq w_0$ and

$$V(t_k^+,U(t_k^+)) = V(t_k^+,I_k(U(t_k))) \leq \psi_k(V(t,U(t_k))), \quad t = t_k$$

Hence by the Theorem 1.4.3 from the (Lakshmikantham, V., and Leela, S. 1969)

$$V(t,U(t)) \leq r(t),\ \ t \geq t_0.$$

Where $U(t) = U(t,t_0,\chi_0(t_0))$ is any solution of the system (11) existing on $[t_0,\infty)$. □

Now we prove the stability result for impulsive set differential equation involving causal operator with memory.

3.7 Theorem

Assume that
(i) $V : \mathbb{R}_+ \times S(\rho) \to \mathbb{R}_+$, $V \in V_0$,

$$D^+V(t,U) \leq g(t,V(t,U)), \qquad t \neq t_k$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, g satisfies (H_2) and $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$ is non decreasing;
ii) there exists $\rho_0 > 0$ such that $U \in S(\rho_0)$ implies $I_k(U) \in S(\rho)$, for all k and

$$V(t_k^+,I_k(U(t_k))) \leq \psi_0(V(t_k,U(t_k))), \qquad U \in S(\rho)$$

and $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$ is non decreasing, $\psi_k(0) = 0$;
iii)

$$b(D[U,\theta]) \leq V(t,U) \leq a(D[U,\theta]), \qquad (t,U) \in \mathbb{R}_+ \times S(\rho) \tag{16}$$

where $a,b \in K$, then the stability properties of the trivial solution of the impulsive scalar differential equation

$$\begin{cases} w' = g(t,w), \quad t \neq t_k \\ w(t_k^+) = \psi_k(w(t_k)) \quad t = t_k \\ w(t_0^+) = w_0 \geq 0, \quad k = 1,2,3,.... \end{cases} \tag{17}$$

implies the corresponding stability property of trivial solution of impulsive set differential equation involving causal operators with memory (11)

Proof. Let $0 < \epsilon < \rho^* = min(\rho_0,\rho)$ and $t_0 \in \mathbb{R}_+$ be given. Let us suppose that the trivial solution of (17) is stable. Then for given $b(\epsilon) > 0$ there exists a $\delta_1(t_0,\epsilon)$ such that

$$0 \leq w_0 \leq \delta_1 \quad implies \quad w(t,t_0,w_0) < b(\epsilon), \quad t \geq t_0$$

where $w(t,t_0,w_0)$ is any solution of (17) passing through the point (t_0,w_0). Let us suppose that $w_0 = a(D[\chi_0,\theta])$ and choose $\delta_2 = \delta_2(\epsilon)$ such that $a(\delta_2) < \delta_1$. Now consider $\delta = min(\delta_1,\delta_2)$, with this δ, we have to claim that

$$D[\chi_0,\theta] < \delta \ \ then \ \ D[U(t),\theta] < \epsilon,\ \ t \geq t_0$$

where $U(t) = U(t, t_0, \chi_0)$ is any solution of (11) Suppose that it is not true, then there exists a solution $U(t) = U(t, t_0, \chi_0)$ of (11) with $D[\chi_0, \theta] < \delta$ and a $t^* > t_0$ such that $t_k < t^* < t_{k+1}$ for some k satisfying $\epsilon \leq D[U(t^*), \theta]$ and $D[U(t), \theta] < \epsilon$ for $t_0 \leq t \leq t_k$.

Since $0 < \epsilon < \rho_0$ from condition (ii) we have

$$D[U(t_k^+), \theta] = D[I_k(U(t_k)), \theta] < \rho \quad and \quad D[U(t_k), \theta] < \epsilon$$

Hence we can find a t^0 such that $t_k < t^0 \leq t^*$ satisfying

$$\epsilon \leq D[U(t^0), \theta] < \rho$$

Setting m(t) = V(t, U(t)) for $t_0 \leq t \leq t^0$ and using the hypothesis (i), (ii) and from comparison result (3.6)

$$V(t, U(t)) \leq r(t, t_0, a(D[\psi_0, \theta])), \quad t_0 \leq t \leq t^0$$

where $r(t, t_0, w_0)$ is the maximal solution of scalar impulsive differential equation (17) . Now

$$b(\epsilon) \leq b(D[U(t^0), \theta]) \leq V(t^0, U(t^0)) \leq r(t^0, t_0, a(D[\chi_0, \theta])) < b(\epsilon)$$

which is a contradiction. Therefore our assumption is wrong. Hence the trivial solution of (11) is stable.
Let us suppose that $w \equiv 0$ of (17) is asymptotically stable. This implies that $U(t) \equiv \theta$ of (11) is stable. Hence set $\epsilon = b^*$ and $\delta_0^* = \delta(t_0, b^*)$, then by definition of stability we have

$$D_0[\chi_0, \theta] < \delta_0^* \Rightarrow D[U(t), \theta] < b^*, \quad t \geq t_0$$

Now we prove attractivity. Let $0 < \epsilon < b^*$ and $t_0 \in \mathbb{R}_+$ there exists a $\eta = \eta(t_0) > 0$ and $T = T(t_0, \epsilon) > 0$ such that

$$0 \leq w_0 \leq \eta \Longrightarrow w(t, t_0, w_0) < b(\epsilon) \quad for \quad t \geq t_0 + T$$

By proceeding previous argument in stability case, we get

$$V(t, U(t)) \leq r(t, t_0, a(D_0[\chi_0, \theta]))$$

Thus we get $b(D[U(t), \theta]) \leq V(t, U(t)) \leq r(t, t_0, a(D_0[\chi_0, \theta])) \leq b(\epsilon)$,
$for \ \ t \geq t_0 + T$, that implies $D[U(t), \theta] < \epsilon, \ for \ \ t \geq t_0 + T$.
Hence $U(t) \equiv \theta$ is attractive and hence asymptotically stable. □

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