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On Pulse Phenomena involving Hybrid Caputo Fractional Differential Equations with Variable Moments of Impulse

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ABSTRACT: In this paper we have established sufficient conditions that guarantee the absence or presence of pulse phenomena involving hybrid Caputo fractional differential equations of order $q,\ 0 < q < 1$ with variable moments of impulse under the weakened hypothesis of C^q continuity.

1 INTRODUCTION

The theory of fractional differential equations is as old as ordinary differential equations. The first application of the fractional calculus was made by Abel(1802-1829) in the solution of an integral equation that arises in the formulation of the tautochronous problem and many applications in this field are given in [11]. Dynamics of many evolutionary processes from various field as population dynamics, control theory etc. undergo abrupt changes at certain moments of times like earthquake, harvesting, shock etc. These processes are modelled by impulsive differential equations. Many results are established parallel to the theory ordinary differential equations. There are many research articles in fractional differential equations [3], [4], [7], [8], [9]. In the paper [12], solution of a hybrid Caputo fractional differential equation of order $q \in (0, 1)$ with variable moments of impulse, conditions for its existence and continuation are established under the weakened hypothesis of C^q continuity, also the different types of behaviour this solution exhibits were discussed, in which one of them is *pulse phenomena* i.e. the solution may hit the same surface (or barrier) several times. Sufficient conditions that guarantee the absence or presence of pulse phenomena involving ordinary differential equations are established [2]. In this paper we have tried to establish sufficient conditions that guarantee the absence or presence of pulse phenomena involving hybrid Caputo fractional differential equations of order $q,\ 0 < q < 1$ with variable moments of impulse under the weakened hypothesis of C^q continuity. In Section 2, we deal with the preliminaries of fractional differential equations. In Section 3, some examples on pulse phenomena are illustrated. In the Section 4, sufficient conditions that guarantee the absence or presence of pulse phenomena involving hybrid Caputo fractional differential equations of order $q,\ 0 < q < 1$ with variable moments of impulse under the weakened hypothesis of C^q continuity are established. Section 5, concludes the work done in the paper.

2 Preliminaries

In this section, we introduce notations, definitions, results and preliminary facts from [1], [6] that are required in the remainder of this paper.

Definition 2.1. *The Riemann-Liouville fractional integral of order q, where q is a positive real number, of a function x given on the interval $[t_0, T]$, $t_0 \geq 0$ is defined as*

$$D_{t_0}^{-q}x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} x(s)\, ds,\ \ t_0 \leq t \leq T$$

where Γ is the Gamma function.

Definition 2.2. *The Riemann-Liouville fractional derivative of order q, where q is a positive real number, of a function x given on the interval $[t_0, T]$, $t_0 \geq 0$ is defined as*

$$D_{t_0}^{q}x(t) = \frac{1}{\Gamma(p)} \frac{d^n}{dt^n} \left\{ \int_{t_0}^{t} (t-s)^{p-1} x(s)\, ds \right\},\ \ t_0 \leq t \leq T$$

where $n = p + q$ and n is the least positive integer greater than q so that $0 < p \leq 1$.

Definition 2.3. *The Caputo's fractional derivative of order q, where q is a positive real number, of a function x given on the interval $[t_0, T]$, $t_0 \geq 0$ is defined as*

$$^{C}D_{t_0}^{q}x(t) = \frac{1}{\Gamma(p)} \int_{t_0}^{t} (t-s)^{p-1} x^{(n)}(s)\, ds,\ \ t_0 \leq t \leq T$$

where $n = p + q$ and n is the least positive integer greater than q so that $0 < p \leq 1$.

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In particular, the Caputo's fractional derivative of order q, where $0 < q < 1$ is defined as

$$^{C}D^{q}x(t) = \frac{1}{\Gamma(p)}\int_{t_0}^{t}(t-s)^{p-1}x'(s)\,ds,\ \ t_0 \le t \le T.$$

where $p + q = 1$

Definition 2.4. *A function u is said to be C_p continuous i.e., $u \in C_p\left([t_0, t_0 + a], \mathbb{R}\right)$ if and only if $u \in C((t_0, t_0+a], \mathbb{R})$ and $(t-t_0)^p u(t) \in C([t_0, t_0 + a], \mathbb{R})$.*

Definition 2.5. *A function u is said to be C^q continuous i.e., $u \in C^q\left([t_0, T], \mathbb{R}\right)$ if and only if the Caputo derivative $^{C}D^{q}u(t)$ exists and satisfies*

$$^{C}D^{q}u(t) = \frac{1}{\Gamma(p)}\int_{t_0}^{t}(t-s)^{p-1}u'(s)\,ds,\ \ t_0 \le t \le T,$$

where $p + q = 1$.

We observe that $u \in C^q\left([t_0, t_0 + a], \mathbb{R}\right)$, implies that u is continuous and differentiable.

Result 2.1. *$x(t) \in C^q([t_0, t_0 + a], \mathbb{R})$ is solution of the initial value problem*

$$^{C}D^{q}\,x = f(t, x),\ \ x(t_0) = x_0,\ \ 0 < q < 1$$

if and only if it satisfies corresponding Volterra fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\,f(s, x(s))\,ds,\ \ t_0 \le t \le t_0 + a.$$

We then state the following lemma and theorem from [10].

Lemma 2.1. *Let $0 < q < 1$. Consider the Caputo fractional differential equation*

$$^{C}D^{q}_{t_0}u(t) = g(t, u(t)),\ \ t \ge t_0,$$

where $g(t, u) \ge 0$ and $t_0 \in \mathbb{R}$. If the solutions exist and $u(t_0) \ge 0$, then they are nonnegative. Furthermore, if $g(t, u) = \lambda u$ for $\lambda \ge 0$, then the solutions are nondecreasing in t.

Theorem 2.1. *Suppose that $0 < q < 1$ and $^{C}D^{q}_{t_0}v(t) \ge {}^{C}D^{q}_{t_0}w(t)$ on $\mathbb{R}_+$. If $v(t_0) \ge w(t_0)$ then $v(t) \ge w(t)$ on $[t_0, \infty)$.*

We next state the following results from [5], which are essential to serve our purpose.

Lemma 2.2. *Let $m \in C_p[[t_0, T], \mathbb{R}]$. Suppose that for any $t_1 \in (t_0, T]$, we have $m(t_1) = 0$ and $m(t) < 0$ for $t_0 \le t < t_1$, then it follows that*

$$D^{q}m(t_1) \ge 0.$$

Corollary 2.1. *Let $m \in C_p[[t_0, T], \mathbb{R}]$. Suppose that for any $t_1 \in (t_0, T]$, we have $m(t_1) = 0$ and $m(t) > 0$ for $t_0 \le t < t_1$, then it follows that*

$$D^{q}m(t_1) \le 0.$$

Remark 2.1. *Tha main advantage of the Caputo fractional derivative is that the initial conditions for fractional differential equations are the same form as that of ordinary differential equations with positve integer derivatives . Another differtence is that the Caputo fractional derivative of order q, $0 < q < 1$ for a constant c is zero, while the Riemann-Liouville fractional derivative for a constant c is not equals to zero but equals to $D^{q}_{t_0}c = \frac{c(t-t_0)^{-q}}{\Gamma(1-q)}$.*

$$^{C}D^{q}_{t_0}x(t) = D^{q}_{t_0}[x(t) - x(t_0)] = D^{q}_{t_0}x(t) - x(t_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}$$

In particular, if $x(t_0) = 0$, then $^{C}D^{q}_{t_0}x(t) = D^{q}_{t_0}x(t)$.

Corollary 2.2. *Let $m \in C_p[[t_0, T], \mathbb{R}]$. Suppose that for any $t_1 \in (t_0, T]$, we have $m(t_1) = 0$ and $m(t) < 0$ for $t_0 \le t < t_1$, then it follows that*

$$^{C}D^{q}m(t_1) \ge 0.$$

Corollary 2.3. *Let $m \in C_p[[t_0, T], \mathbb{R}]$. Suppose that for any $t_1 \in (t_0, T]$, we have $m(t_1) = 0$ and $m(t) > 0$ for $t_0 \le t < t_1$, then it follows that*

$$^{C}D^{q}m(t_1) \le 0.$$

3 Examples

Consider the initial value problem (IVP) for Caputo fractional differential equation of order $q,\ 0 < q < 1$ with variable moments of impulse given by

$$\left.\begin{array}{rcl} {}^C D^q x & = & f(t,x),\ t \neq \tau_k(x), \\ x(t^+) & = & x(t) + I_k(x),\ t = \tau_k(x), \\ x(t_0^+) & = & x_0,\ t_0 \geq 0. \end{array}\right\} \tag{1}$$

where $f \in C[I \times \Omega, \mathbb{R}]$, $I = [t_0, T]$, $t_0 \geq 0$, $\Omega \subset \mathbb{R}$ being an open set, $I_k(x) \in C[\Omega, \mathbb{R}]$, and $\tau_k(x) \in C^q[\Omega, (0, \infty)]$ and $S_k : t = \tau_k(x)$ are linear surfaces of the form $t = \lambda x + k$ so that $\tau_k(x) = \lambda x + k$ where $\lambda \in \mathbb{R} - \{0\}, k \in \mathbb{Z}^+$. It then follows that $\tau_k(x) < \tau_{k+1}(x)$, for every $k \in \mathbb{Z}^+$ on $\Omega,\ x(t^+) = \lim_{h \to 0} x(t+h),\ x(t^-) = \lim_{h \to 0} x(t-h),\ \Delta x = x(t^+) - x(t)$.

Definition 3.1.

A function $x : [t_0, t_0 + a) \to \mathbb{R}$, $t_0 \geq 0, a > 0$, is said to be a solution of (1) if
(i) $x(t_0^+) = x_0$ and $(t, x(t)) \in D$ for $t \in [t_0, t_0 + a)$,
(ii) $x(t) \in C^q([t_0, t_0 + a), \mathbb{R})$, ${}^c D^q x(t)$ is continuous, and $x(t)$ satisfies ${}^c D^q x = f(t, x)$ for $t \in [t_0, t_0 + a)$ and $t \neq \tau_k(x(t))$,
(iii) if $t \in [t_0, t_0 + a)$ and $t = \tau_k(x(t))$, then $x(t^+) = x(t) + I_k(x(t))$, and at such t's we always assume that $x(t)$ is left continuous and $s \neq \tau_j(x(s))$ for any j, $t < s < t + \delta$, for some $\delta > 0$.

Whenever $t_0 \neq \tau_k(x_0)$ for any k, we mean the initial condition $x(t_0^+) = x_0$ in the usual sense, that is, $x(t_0) = x_0$. If $t_0 = \tau_k(x_0)$ for some k then $x(t_0^+) = x_0$, which, in general, is natural for the system (1), since (t_0, x_0) may be such that $t_0 = \tau_k(x_0)$
We have illustrated pulse phenomena through examples where the solution hit the same surface finite or infinite number of times.

Example 3.1.

Consider the Caputo fractional differential equation with variable moments of impulse

$$\left.\begin{array}{rcl} {}^C D^q x & = & 0,\ t \neq \tau_k(x), \\ \Delta x & = & |x - 5|,\ t = \tau_k(x)\ . \\ x(0^+) & = & 1, \end{array}\right\} \tag{2}$$

where $\tau_k(x) = x + k,\ k \in \mathbb{Z}^+$.
The solution of (2) starting at $(0, 1)$ hits the surface S_1 at $(2, 1)$ and then at $(6, 5)$ experiencing the pulse phenomena. In this case the solution of (2) hits the surface S_1 a finite number of times, see figure 1.

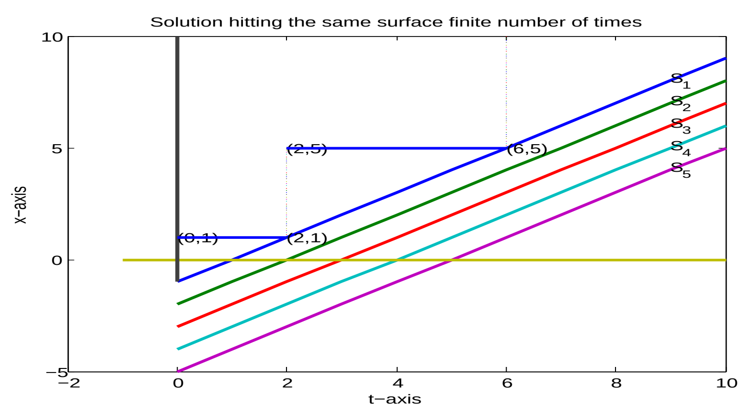

Figure 1:

Example 3.2.

Consider the Caputo fractional differential equation with variable moments of impulse

$$\left.\begin{array}{rcl} {}^C D^q x & = & 0,\ t \neq \tau_k(x), \\ \Delta x & = & x^2 - x,\ t = \tau_k(x)\ . \\ x(0^+) & = & 0.9, \end{array}\right\} \tag{3}$$

where $\tau_k(x) = -x + k,\ k \in \mathbb{Z}^+$.
The solution of (3) starting at $(0, 0.9)$ hits the surface S_1 an infinite number of times experiencing the pulse phenomena, see figure 2.

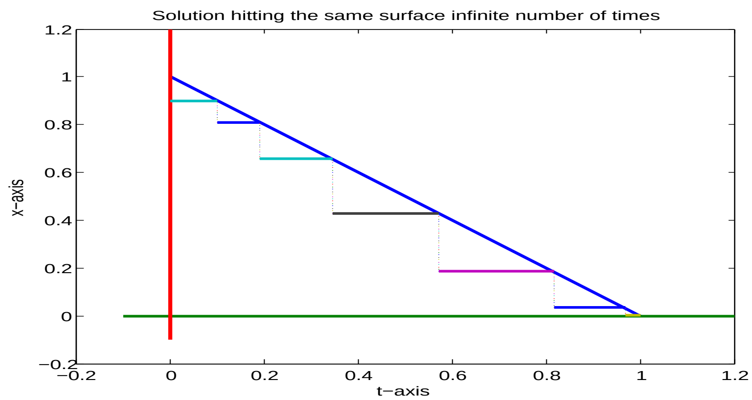

Figure 2:

4 Pulse phenomena

We now propose to give a simple set of sufficient conditions for any solution to meet each surface exactly once and shows the interplay between the functions f, τ_k and I_k. In the rest of the section we shall assume that solutions of (1) exists for $t \geq t_0$ and it is C_p continuous.

Theorem 4.1. *Assume that*
(i) $f \in C[\,[t_0, T] \times \Omega, \mathbb{R}],\ t_0 \geq 0, I_k \in C[\Omega, \mathbb{R}],\ \tau_k \in C^q[\Omega, (0, \infty)],\ \tau_k(x)$ *is linear and bounded, and* $\tau_k(x) < \tau_{k+1}(x)$ *for each* k.

(ii)(a) $\dfrac{\partial \tau_k(x)}{\partial x} f(t, x) < \dfrac{(t - \tilde{t})^p}{\Gamma(p+1)},\ whenever\ t = \tau_k(x(t, \tilde{t}, \tilde{x}))$.

(b) $\left(\dfrac{\partial \tau_k}{\partial x}(x + sI_k(x))\right) I_k(x) < 0$, *and*

(c) $\left(\dfrac{\partial \tau_k}{\partial x}(x + sI_{k-1}(x))\right) I_{k-1}(x) \geq 0,\ 0 \leq s \leq 1,\ x + I_k(x) \in \Omega$
whenever $x \in \Omega$.

Then every solution $x(t) = x(t, t_0, x_0)$ *of IVP* (1) *such that* $0 \leq t_0 < \tau_1(x_0)$ *meets each surface* S_k *exactly once.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of IVP (1) such that $0 \leq t_0 < \tau_1(x_0)$. Since $\tau_1(x)$ is bounded and continuous on Ω, there is a unique $t_1 > t_0$ such that

$$t_1 = \tau_1(x(t_1)) \text{ and } t < \tau_1(x(t)) \text{ for } t < t_1. \tag{4}$$

Therefore $x(t)$ hits the surface S_1 at $t = t_1$. Let $x_k = x(t_k),\ x_k^+ = x_k + I_k(x_k)$. By condition (ii)(c), for each $x \in \Omega$, $x + I_i(x) \in \Omega$. Also each $\tau_i(x)$ is differentiable in Ω.

Now, $\tau_i(x + I_i(x)) - \tau_i(x) = \displaystyle\int_0^1 \left(\frac{\partial \tau_i}{\partial x}(x + sI_i(x))\right) I_i(x)ds < 0$, by condition (ii)(b). So we have $\tau_i(x + I_i(x)) < \tau_i(x)$ for any $x \in \Omega$ and $i \geq 1$. This implies that

$$\tau_1(x_1^+) = \tau_1(x_1 + I_1(x_1)) < \tau_1(x_1) = t_1. \tag{5}$$

Since $\tau_k(x) < \tau_{k+1}(x)$ for each k, we have

$$t_1 = \tau_1(x_1) < \tau_2(x_1). \tag{6}$$

Now, $\tau_i(x + I_{i-1}(x)) - \tau_i(x) = \displaystyle\int_0^1 \left(\frac{\partial \tau_i}{\partial x}(x + sI_{i-1}(x))\right) I_{i-1}(x)ds \geq 0$, by condition (ii)(c). So we have $\tau_i(x + I_{i-1}(x)) \geq \tau_i(x)$ for any $x \in \Omega$ and $i \geq 2$.
This implies that

$$\tau_2(x_1) \leq \tau_2(x_1 + I_1(x_1)) = \tau_2(x_1^+). \tag{7}$$

From (5), (6) and (7) it follows that

$$\tau_1(x_1^+) < t_1 < \tau_2(x_1^+). \tag{8}$$

Since $\tau_1(x_1^+) < t_1 < \tau_2(x_1^+)$, by proceeding as before, we can find a unique $t_2 > t_1$ such that $t_2 = \tau_2(x(t_2, t_1, x_1^+))$ and $t < \tau_2\,(x(t, t_1, x_1^+))$ for $t_1 < t < t_2$. Now we will show that $x(t)$ hits the surface $S_1 : t = \tau_1(x)$ only once.

Set $m(t) = t - \tau_1(x(t, t_1, x_1^+)),\ t \geq t_1$. From (5), we have $m(t_1) = t_1 - \tau_1(x_1^+) > 0$. We calim that $m(t) > 0$ for all $t \geq t_1$. If not, there exists a $t^* > t_1$ such that $m(t^*) = 0$ i.e $t^* = \tau_1(x(t^*, t_1, x_1^+)),$ and $m(t) > 0,\ t_1 \leq t < t^*$. By Corollary (2.2), this implies that ${}^C D^q_{t_1} m(t^*) \leq 0$. But

$$\begin{aligned}
{}^c D^q_{t_1}\{m(t)\} &= {}^C D^q_{t_1}\{t - \tau_1(x(t, t_1, x_1^+))\} = {}^C D^q_{t_1}\{t\} - {}^C D^q_{t_1}\{\tau_1(x(t, t_1, x_1^+)\} \\
&= \frac{1}{\Gamma(p)} \int_{t_1}^{t} (t-s)^{p-1} ds - \frac{1}{\Gamma(p)} \int_{t_1}^{t} (t-s)^{p-1} \frac{\partial \tau_1(x)}{\partial x} x'(s, t_1, x_1^+) ds \\
&= -\left[\frac{(t-s)^p}{p\,\Gamma(p)}\right]_{t_1}^{t} - \frac{\partial \tau_1(x)}{\partial x}.f(t, x(t, t_1, x_1^+)) \\
&= \left\{\frac{(t-t_1)^p}{\Gamma(p+1)} - \frac{\partial \tau_1(x)}{\partial x}.f(t, x(t, t_1, x_1^+))\right\},\ \forall\, t_1 \leq t \leq t^*.
\end{aligned}$$

By condition (ii)(a), ${}^C D^q_{t_1} m(t^*) > 0$, which is a contradiction. Thus $m(t) > 0,$ for all $t \geq t_1$ that is $t > \tau_1(x(t)),\ \forall\, t \geq t_1$ and therefore $x(t)$ hits the surface S_1 only once. Therefore, $x(t)$ meets the second surface S_2 first at $t = t_2$ after it hitting the first surface S_1 at $t = t_1$. Also it follows that the solution $x(t)$ hits the surface S_1 exactly once in $[t_0, t_2]$ and also in $[t_0, T]$.

Setting again $x_2 = x(t_2, t_1, x_1^+),\ x_2^+ = x_2 + I_2(x_2)$ and using condition (ii), we can conclude that $\tau_2(x_2^+) < t_2 < \tau_3(x_2^+)$. By arguing as before, we can find a unique $t_3 > t_2$ such that $x(t)$ meets S_3 first at $t = t_3$ after it hitting the surface S_2 at $t = t_2$. This implies that the solution $x(t)$ hits the surface S_2 exactly once in $[t_1, t_3]$ and also in $[t_0, T]$. By repeating this process, one can prove the stated claim and therefore the proof is complete. □

We shall now obtain conditions for pulse phenomena to occur. First, we consider a simple situation where we have only one surface.

Theorem 4.2. *Assume that*

(i) $f \in C[\,[t_0, T] \times \Omega, \mathbb{R}],\ t_0 \geq 0, I \in C[\Omega, \mathbb{R}],\ \tau \in C^q[\Omega, (0, \infty)],\ \tau(x)$ *is linear and bounded.*

(ii) $x + I(x) \in \Omega$ *for* $x \in \Omega$ *and* $\dfrac{\partial \tau}{\partial x}(x + sI(x))I(x) > 0,\ 0 \leq s \leq 1.$

Then, every solution $x(t) = x(t, t_0, x_0)$ *of*

$$\left.\begin{aligned}
{}^C D^q x &= f(t, x),\ t \neq \tau(x), \\
x(t^+) &= x(t) + I(x(t)),\ t = \tau(x), \\
x(t_0^+) &= x_0,\ t_0 \geq 0\,.
\end{aligned}\right\} \tag{9}$$

such that $0 \leq t_0 < \tau(x_0)$ *meets the surface* $S : t = \tau(x)$ *several times.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (9) such that $t_0 < \tau(x_0)$. Since $\tau(x)$ is bounded and continuous on Ω, there exists $t_1 > t_0$ such that $t_1 = \tau(x(t_1))$. This implies that $x(t)$ hits the surface S at $t = t_1$.

Let $x_1 = x(t_1),\ x_1^+ = x_1 + I(x_1))$. By condition (ii), we have for each $x \in \Omega$, $x + I(x) \in \Omega$ and $\tau(x)$ is differentiable in Ω.

Now, $\tau(x + I(x)) - \tau(x) = \displaystyle\int_0^1 \left(\frac{\partial \tau}{\partial x}(x + sI(x))\right) I(x)\, ds > 0$, by condition (ii). So we have $\tau(x + I(x)) > \tau(x)$ for any $x \in \Omega$. This implies that

$$\tau(x_1^+) = \tau(x_1 + I(x_1)) > \tau(x_1) = t_1. \tag{10}$$

Let $x(t) = x(t, t_1, x_1^+)$ be any solution of (9) starting at (t_1, x_1^+). Since $\tau(x)$ is bounded, continuous on Ω, and $t_1 < \tau(x_1^+)$, there exists a $t_2 > t_1$ such that $t_2 = \tau(x(t_2, t_1, x_1^+))$. This implies that $x(t)$ hits the same surface S second time at $t = t_2$. This process can be continued as long as the solution $x(t)$ remain in Ω and therefore the proof is complete. □

The next result offers conditions for any solution to hit a given surface S_i several times.

Theorem 4.3. *Assume that*
(i) $f \in C[[t_0,T]\times\Omega,\mathbb{R}],\ t_0\geq 0, I_k\in C[\Omega,\mathbb{R}],\ \tau_k\in C^q[\Omega,(0,\infty)],\ \tau_k(x)$ *is linear and bounded,* $\tau_k(x)<\tau_{k+1}(x)$ *for each* k.

(ii) (a) $\dfrac{\partial\tau_{i-1}(x)}{\partial x}\,f(t,x)<\dfrac{(t-\tilde t\,)^p}{\Gamma(p+1)},$ *whenever* $t=\tau_{i-1}(x(t,\tilde t,\tilde x))$.

(b) $x+I_i(x)\in\Omega$ *for* $x\in\Omega$, $\left[\dfrac{\partial\tau_i}{\partial x}(x+sI_i(x))\right]I_i(x)>0$, *and*

(c) $\left[\dfrac{\partial\tau_{i-1}}{\partial x}(x+sI_i(x))\right]I_i(x)\leq 0$ *for* $0\leq s\leq 1$.

Then, every solution $x(t)=x(t,t_0,x_0)$ *of IVP (1) such that* $\tau_{i-1}(x_0)<t_0<\tau_i(x_0)$ *meets the surface* S_i *several times.*

Proof. Let $x(t)=x(t,t_0,x_0)$ be any solution of IVP (1) such that

$$\tau_{i-1}(x_0)<t_0<\tau_i(x_0). \tag{11}$$

Since $t_0<\tau_i(x_0)$, $\tau_i(x)$ is bounded and continuous on Ω, there exists a unique $t_1>t_0$ such that $t_1=\tau_i(x(t_1))$ and $t<\tau_i(x(t))$ for all $t_0<t<t_1$. That is the solution $x(t)=x(t,t_0,x_0)$ starting at (t_0,x_0) hit the surface S_i first at $t=t_1$. Now we will show that the solution $x(t)=x(t,t_0,x_0)$ starting at (t_0,x_0) does not hit the surface S_{i-1} in $[t_0,t_1]$. Set $T(t)=t-\tau_{i-1}(x(t,t_0,x_0)),\ t\geq t_0$. From (11), we have $T(t_0)=t_0-\tau_{i-1}(x_0)>0$. We claim that $T(t)>0$ for all $t\geq t_0$. If not, there exists a $t^*>t_0$ such that $T(t^*)=0$ i.e $t^*=\tau_{i-1}(x(t^*,t_0,x_0))$ and $T(t)>0,\ t_0\leq t<t^*$. By Corollary (2.2), this implies that ${}^CD^q_{t_0}T(t^*)\leq 0$. But

$$\begin{aligned}{}^CD^q_{t_0}\{T(t)\} &= {}^CD^q_{t_0}\{t-\tau_{i-1}(x(t,t_0,x_0))\}={}^CD^q_{t_0}\{t\}-{}^CD^q_{t_0}\{\tau_{i-1}(x(t,t_0,x_0)\}\\ &= \frac{(t-t_0)^p}{\Gamma(p+1)}-\frac{\partial\tau_{i-1}(x)}{\partial x}.f(t,x(t,t_0,x_0)),\ \forall\, t_0\leq t\leq t^*.\end{aligned}$$

By condition (ii)(a), ${}^CD^q_{t_0}T(t^*)>0$, which is a contradiction. Thus $T(t)>0,$ for all $t\geq t_0$. This implies that

$$\tau_{i-1}(x(t,t_0,x_0))<t \text{ for } t\in[t_0,t_1]. \tag{12}$$

Hence the solution $x(t)=x(t,t_0,x_0)$ starting at (t_0,x_0) does not hit the surface S_{i-1} in $[t_0,t_1]$. Hence $x(t)$ meets the surface S_i at $t=t_1$ before hitting any other surface. Let $x_1=x(t_1)$ and $x_1^+=x_1+I_i(x_1)$. By condition (ii)(b), for each $x\in\Omega$, $x+I_j(x)\in\Omega$. Also each $\tau_j(x)$ is differentiable in Ω.

Now, $\tau_j(x+I_j(x))-\tau_j(x)=\displaystyle\int_0^1\left(\frac{\partial\,\tau_j}{\partial x}\,(x+sI_j(x))\right)I_j(x)ds>0$, by condition (ii)(b). So we have $\tau_j(x+I_j(x))>\tau_j(x)$ for any $x\in\Omega$ and $j\geq 1$.
This implies that

$$t_1=\tau_i(x_1)<\tau_i(x_1+I_i(x_1))=\tau_i(x_1^+). \tag{13}$$

Similarly, $\tau_{i-1}(x+I_i(x))-\tau_{i-1}(x)=\displaystyle\int_0^1\left(\frac{\partial\,\tau_{i-1}}{\partial x}\,(x+sI_i(x))\right)I_i(x)ds\leq 0$, by condition (ii)(c). So we have $\tau_{i-1}(x+I_i(x))\leq\tau_{i-1}(x)$ for any $x\in\Omega$.
This implies that

$$\tau_{i-1}(x_1^+)\leq\tau_{i-1}(x_1). \tag{14}$$

Since $\tau_k(x)<\tau_{k+1}(x)$ for each $k,\ \forall\,x\in\Omega$ we have

$$\tau_{i-1}(x_1)<\tau_i(x_1)=t_1. \tag{15}$$

From (13), (14) and (15) it follows that

$$\tau_{i-1}(x_1^+)<t_1<\tau_i(x_1^+). \tag{16}$$

Let $x(t)=x(t,t_1,x_1^+)$ be any solution of IVP (1)) starting at (t_1,x_1^+). Since $t_1<\tau_i(x_1^+)$, $\tau_i(x)$ is bounded and continuous on Ω, by proceeding as earlier, there exists a $t_2>t_1$ such that $t_2=\tau_i(x(t_2))$ and $t<\tau_i(x(t))$ for all $t_1<t<t_2$. This shows that every solution $x(t)$ hits the surface S_j at least twice. We can now repeat the same argument as long as the solution remains in Ω and therefore the proof is complete. □

The pulse phenomena can occur in many complicated ways. We shall now give a typical result in that direction.

Theorem 4.4. *Assume that*
(i) $f \in C[[t_0, T] \times \Omega, \mathbb{R}],\ t_0 \geq 0,\ I_k \in C[\Omega, \mathbb{R}], \tau_k \in C^q[\Omega, (0, \infty)],\ \tau_k(x)$ *is linear and bounded,* $\tau_k(x) < \tau_{k+1}(x)$ *for every* k.

(ii) $\dfrac{\partial \tau_k(x)}{\partial x} f(t,x) < \dfrac{(t-\tilde{t})^p}{\Gamma(p+1)},$ *whenever* $t = \tau_k(x(t, \tilde{t}, \tilde{x}))$.
(iii) $x + I_k(x) \in \Omega$ *for* $x \in \Omega$ *and every* k, *and for some fixed* $k = k_0$,

$(b_1)\left(\dfrac{\partial \tau_{k_0}}{\partial x}(x + sI_j(x))\right) I_j(x) \geq 0,\ \tau_j(x) > \tau_{k_0-1}(x + I_j(x))$ *for all* $j < k_0$, *where* $0 \leq s \leq 1$.

$(b_2)\left(\dfrac{\partial \tau_{k_0-1}}{\partial x}(x + sI_j(x))\right) I_j(x) \leq 0,\ \tau_j(x) < \tau_{k_0}(x + I_j(x))$ *for all* $j > k_0$, *where* $0 \leq s \leq 1$.

Then, every solution $x(t) = x(t, t_0, x_0)$ *of IVP (1) meets the surface* S_{k_0} *several times.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of IVP (1) starting at (t_0, x_0). Since each $\tau_k(x)$ is bounded and continuous on Ω, there exists a $t_1 > t_0$ such that $t_1 = \tau_j(x(t_1))$ for some j. That is, $x(t)$ meets the surface S_j at $t = t_1$. We need to consider two cases, namely $j < k_0$ or $j > k_0$.
Case (A). Let $j < k_0$.
Let $x_1 = x(t_1)$ so that $x_1^+ = x_1 + I_j(x_1)$. By arguing as earlier, by condition $(iii)(b_1)$, it follows that

$$\tau_{k_0}(x_1) \leq \tau_{k_0}(x_1 + I_j(x_1)) = \tau_{k_0}(x_1^+). \tag{17}$$

Since $j < k_0$ and $\tau_k(x) < \tau_{k+1}(x)$ for every k on Ω we have

$$t_1 = \tau_j(x_1) < \tau_{k_0}(x_1). \tag{18}$$

From the condition (iii)(b_1) it follows that

$$t_1 = \tau_j(x_1) > \tau_{k_0-1}(x_1 + I_j(x_1)) = \tau_{k_0-1}(x_1^+). \tag{19}$$

From (17), (18) and (19) we have

$$\tau_{k_{0}-1}(x_1^+) < t_1 < \tau_{k_0}(x_1^+). \tag{20}$$

Let $x(t) = x(t, t_1, x_1^+)$ be the solution of IVP (1) starting at (t_1, x_1^+). Since $\tau_{k_0}(x)$ is bounded and continuous on Ω, there exists a $t_2 > t_1$ such that $t_2 = \tau_{k_0}(x(t_2))$. That is, $x(t)$ meets the surface S_{k_0} at $t = t_2$.
Now we will show that the solution $x(t) = x(t, t_1, x_1^+)$ starting at (t_1, x_1^+) does not hit the surface S_{k_0-1} in $[t_1, t_2]$.

Set $T(t) = t - \tau_{k_0-1}(x(t, t_1, x_1^+)),\ t \geq t_1$. From (20), we have $T(t_1) = t_1 - \tau_{k_0-1}(x_1^+) > 0$. We claim that $T(t) > 0$ for all $t \geq t_1$. If not, there exists a $t^* > t_1$ such that $T(t^*) = 0$ i.e. $t^* = \tau_{k_0-1}(x(t^*, t_1, x_1^+))$ and $T(t) > 0,\ t_0 \leq t < t^*$. By Corollary (2.3), this implies that ${}^C D_{t_1}^q T(t^*) \leq 0$. But

$$\begin{aligned} {}^C D_{t_1}^q \{T(t)\} &= {}^C D_{t_1}^q \{t - \tau_{k_0-1}(x(t, t_1, x_1))\} \\ &= {}^C D_{t_1}^q \{t\} - {}^C D_{t_1}^q \{\tau_{k_0-1}(x(t, t_1, x_1)\} \\ &= \frac{(t-t_1)^p}{\Gamma(p+1)} - \frac{\partial \tau_{k_0-1}(x)}{\partial x} . f(t, x(t, t_1, x_1)),\ \forall\ t_1 \leq t \leq t^*. \end{aligned}$$

By condition (ii), ${}^C D_{t_1}^q T(t^*) > 0$, which is a contradiction. Thus $T(t) > 0$, for all $t \geq t_1$ that is $t > \tau_{k_0-1}(x(t)),\ \forall\, t \geq t_1$. This implies that

$$\tau_{k_0-1}(x(t, t_1, x_1)) < t \text{ for } t \in [t_1, t_2]. \tag{21}$$

This implies that the solution $x(t) = (t, t_1, x_1)$ starting at (t_1, x_1) does not hit the surface S_{k_0-1} in $[t_1, t_2]$. Hence $x(t)$ meets the surface S_{k_0} at $t = t_2$ before hitting any other surface.

Case (B). Let $j > k_0$.
Let $x_1 = x(t_1)$ so that $x_1^+ = x_1 + I_j(x_1)$. By arguing as earlier, by condition $(iii)(b_2)$, it follows that

$$\tau_{k_0-1}(x_1) \geq \tau_{k_0-1}(x_1 + I_j(x_1)) = \tau_{k_0-1}(x_1^+). \tag{22}$$

Since $j > k_0$ and $\tau_k(x) < \tau_{k+1}(x)$ for every k, on Ω we have

$$t_1 = \tau_j(x_1) > \tau_{k_0}(x_1) > \tau_{k_0-1}(x_1). \tag{23}$$

From the condition (iii)(b_2), it follows that

$$t_1 = \tau_j(x_1) < \tau_{k_0}(x_1 + I_j(x_1)) = \tau_{k_0}(x_1^+). \tag{24}$$

From (22), (23) and (24) we have

$$\tau_{k_0-1}(x_1^+) < t_1 < \tau_{k_0}(x_1^+). \tag{25}$$

Let $x(t) = x(t, t_1, x_1^+)$ be the solution of IVP (1) starting at (t_1, x_1^+). Since $\tau_{k_0}(x)$ is bounded and continuous on Ω, there exists a $t_2 > t_1$ such that $t_2 = \tau_{k_0}(x(t_2))$. That is, $x(t)$ meets the surface S_{k_0} at $t = t_2$. By setting $T(t) = t - \tau_{k_0-1}(x(t, t_1, x_1))$, and using the fact (ii), (25) and the Corollary (2.3), as proceeding in the Case A, we can show that $T(t) > 0, \forall\, t \geq t_1$. This implies that

$$\tau_{k_0-1}(x(t, t_1, x_1)) < t \text{ for } t \in [t_1, t_2]. \tag{26}$$

This implies that the solution $x(t) = (t, t_1, x_1)$ starting at (t_1, x_1) does not hit the surface S_{k_0-1} in $[t_1, t_2]$.

If $x(t)$ hits the surface S_{k_0} several times after $t = t_2$, we are done, If not, $x(t)$ encounters some surface $S_j,\ i \neq k_0$ at $t_3 > t_2$, because $\tau_k(x)$ is bounded on Ω for every k. By Arguing as before, we can show that there exists a $t_4 > t_3$ at which $x(t)$ meets S_{k_0} again. This process can be continued as long as the solutions exist and therefore, the desired result follows proving the theorem. □

Now we are going to give sufficient conditions for not hitting a fixed surface S_j .

Theorem 4.5. *Assume that*
(i) $f \in C[\,[t_0, T] \times \Omega, \mathbb{R}],\ t_0 \geq 0, I_k \in C[\Omega, \mathbb{R}],\ \tau_k \in C^q[\Omega, (0, \infty)],\ \tau_k(x)$ *is linear and bounded,* $\tau_k(x) < \tau_{k+1}(x)$ *for each* k.

(ii) for any fixed $j \geq 3$, $\dfrac{\partial \tau_j(x)}{\partial x} f(t, x) < \dfrac{(t - \tilde{t}\,)^p}{\Gamma(p+1)}$,
$whenever\ t = \tau_j(x(t, \tilde{t}, \tilde{x}))$;
(iii) $\tau_j(x + I_{j-1}(x)) < \tau_{j-1}(x),\ x + I_{j-1}(x) \in \Omega,\ \forall x \in \Omega$

Then, every solution $x(t) = x(t, t_0, x_0)$ *of IVP (1) such that* $\tau_{j-2}(x_0) < t_0 < \tau_{j-1}(x_0)$ *does not hit the surface* $S_j : t = \tau_j(x)$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of IVP (1) such that

$$\tau_{j-2}(x_0) < t_0 < \tau_{j-1}(x_0). \tag{27}$$

Since $t_0 < \tau_{j-1}(x_0)$, $\tau_{j-1}(x)$ is bounded and continuous on Ω, there exists a unique $t_1 > t_0$ such that

$$t_1 = \tau_{j-1}(x(t_1)) \text{ and } t < \tau_{j-1}(x(t)) \text{ for all } t_0 \leq t < t_1. \tag{28}$$

That is the solution $x(t) = x(t, t_0, x_0)$ starting at (t_0, x_0) hit the surface S_{j-1} at $t = t_1$.
Since $\tau_k(x) < \tau_{k+1}(x)$ for every k, we have

$$\tau_{j-1}(x(t)) < \tau_j(x(t)). \tag{29}$$

From (28), (29) we have

$$t < \tau_j(x(t)), \text{ for all } t_0 \leq t < t_1. \tag{30}$$

This implies that the solution $x(t)$ does not hit the surface $S_j : t = \tau_j(x)$ in $[t_0, t_1)$. Let $x_1 = x(t_1),\ x_1^+ = x_1 + I_{j-1}(x_1)$.
By condition(iii), we have

$$t_1 = \tau_{j-1}(x_1) > \tau_j(x_1^+). \tag{31}$$

Also we have from condition(ii), and the relation (31), we have

$$t > \tau_j(x(t, t_1, x_1^+)),\ \forall\, t \geq t_1. \tag{32}$$

This implies that the solution $x(t)$ does not hit the surface $S_j : t = \tau_j(x)$ in $[t_1, T]$.

From (30), (32) we have $t \neq \tau_j(x(t))$ in $[t_0, T]$. That is the solution $x(t)$ does not hit the surface S_j in $[t_0, T]$ and the proof is complete. □

5 Conclusion

In this paper we have studied pulse phenomena involving hybrid Caputo fractional differential equations of order $q \in (0, 1)$ with variable moments of impulse and have shown by examples the potential it has for further work.

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