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Trigonometric Approximation of Signals (Functions) belonging to the $W(L^r, \xi(t)), (r \geq 1)$-class by (E, q) $(q > 0)$ - means of the conjugate series of its Fourier series

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ABSTRACT: Positive approximation processes play an important role in Approximation Theory and appear in a very natural way dealing with approximation of continuous functions, especially one, which requires further qualitative properties such as monotonicity, convexity and shape preservation and so on. Analysis of signals or time functions is of great importance, because it conveys information or attributes of some phenomenon. The engineers and scientists use properties of Fourier approximation for designing digital filters. Various investigators such as Khan ([1]-[5]), Chandra [6, 7], Leindler [8], Mishra et al. [9], Mishra [10], Mittal et al. [12], Mittal, Rhoades and Mishra [13], Mittal and Mishra [14], Rhoades et al. [15] have determined the degree of approximation of $2\pi-$periodic signals (functions) belonging to various classes $Lip\ \alpha$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ and $W(L_r, \xi(t))$ of functions through trigonometric Fourier approximation (TFA) using different summability matrices with monotone rows. Recently, Mittal et al. [16], Mishra [10], Mishra and Mishra [17] have obtained the degree of approximation of signals belonging to $Lip(\alpha, r)$-class by general summability matrix, which generalizes the results of Leindler [8] and some of the results of Chandra [7] by dropping monotonicity on the elements of the matrix rows (that is, weakening the conditions on the filter, we improve the quality of digital filter). In this paper, a theorem concerning the degree of approximation of the conjugate of a signal (function) f belonging to $W(L^r, \xi(t)), (r \geq 1)$- class by (E, q) summability of conjugate series of its Fourier series has been established which in turn generalizes the results of Chandra [7], Shukla [21] and Mishra et al. [11].

Keywords: Lebesgue integral, Conjugate Fourier series, weighted $W(L^r, \xi(t)), (r \geq 1)$- class, Degree of approximation, Euler (E, q) means.

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1 INTRODUCTION

The approximation of functions by algebraic polynomials, trigonometric polynomials, and splines, is not only an important topic of mathematical studies, but also provides powerful mathematical tools to such application areas as data representation, signal processing, nonparametric time-series analysis, computer-aided geometric design, numerical analysis, and solutions of differential equations. This mathematics of approximation is an introduction to the mathematical analysis of such approximation, with a strong emphasis on explicit approximation formulations, corresponding error bounds and convergence results, as well as applications in quadrature.

A mathematically rigorous approach is adopted throughout, and, apart from an assumed prerequisite knowledge of advanced calculus and linear algebra, the presentation of the material is self-contained. The theory of approximation is a very extensive field and the study of the theory of trigonometric approximation is of great mathematical interest and of great practical importance. Broadly speaking, Signals are treated as functions of one variable and images are represented by functions of two variables. The study of these concepts is directly related to the emerging area of information technology. Khan ([1]- [4]) and Mittal, Rhoades and Mishra [13] have initiated the studies of error estimates $E_n(f)$ through trigonometric Fourier approximation (TFA) using different summability matrices. Chandra [7] has studied the degree of approximation, of a signal (function) belonging to $Lip(\alpha)$ - class by (E, q) means, $q > 0$.

Generalizing the result of Chandra [7], very interesting result has been proved by Shukla [21] for the signals (functions) of $Lip(\alpha, r)$ - class through trigonometric Fourier approximation applying (E, q) $(q > 0)$ summability matrix.

Let $\sum_{n=0}^{\infty} u_n$ be a given series with sequence of its partial sums $\{s_n\}$.

The (E, q) transform is defined as the n^{th} partial sum of (E, q) summability and we denote it by E_n^q.

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If

$$E_n^q = \frac{1}{(q+1)^n}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k} s_k \to s, as\, n \to \infty, \tag{1.1}$$

then the series $\sum_{n=0}^{\infty} u_n$ is summable (E, q) to 's'[18].
A signal (function) $f \in Lip\,\alpha$, if

$$f(x+t) - f(x) = O(|t^{\alpha}|)\ for\ 0 < \alpha \le 1, t > 0 \tag{1.2}$$

and $f \in Lip(\alpha, r),\, for\ 0 \le x \le 2\pi$ [1], if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O\Big(|t|^{\alpha}\Big),\ 0 < \alpha \le 1,\ r \ge 1,\ t > 0. \tag{1.3}$$

Given a positive increasing function $\xi(t)$, $f \in Lip(\xi(t), r)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O\Big(\xi(t)\Big),\, r \ge 1,\ t > 0. \tag{1.4}$$

$f \in W(L_r, \xi(t))$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r sin^{\beta r}\,(x) dx\right)^{\frac{1}{r}} = O\Big(\xi(t)\Big).$$

We define the weighted class as follows

$$\omega_r(t; f) = \left(\int_0^{2\pi} |f(x+t) - f(x)|^r sin^{\beta r}\Big(x/2\Big) dx\right)^{\frac{1}{r}} = O\Big(\xi(t)\Big),\, \beta \ge 0,\ t > 0\ \ [4].$$

If $\beta = 0$, then the weighted class $W(L_r, \xi(t))$ coincides with the class $Lip(\xi(t), r)$, we observe that

$$W(L_r, \xi(t)) \xrightarrow{\beta=0} Lip(\xi(t), r) \xrightarrow{\xi(t)=t^{\alpha}} Lip(\alpha, r) \xrightarrow{r\to\infty} Lip\ \alpha,\ \text{ for } 0 < \alpha \le 1,\ r \ge 1,\ t > 0. \tag{1.5}$$

The L_∞-norm of a signal $f : R \to R$ is defined by $\|f\|_\infty = \sup\{|f(x)| : x \in R\}$.
The L_r- norm of a signal is defined by

$$\|f\|_r = \left(\int_0^{2\pi} \frac{1}{2\pi}|f(x)|^r dx\right)^{\frac{1}{r}},\ (1 \le r < \infty). \tag{1.6}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of order n under sup norm $\|\|_\infty$ is defined by Zygmund [19].

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\}.$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_n \|f(x) - t_n(f; x)\|_r \tag{1.7}$$

in terms of n, where $t_n(f; x)$ is a trigonometric polynomial of degree (order) n.
This method of approximation is called Trigonometric Fourier Approximation (TFA) [13].
Let $f(x)$ be a $2\pi-$ periodic signal (function) and Lebesgue integrable. The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos\, nx + b_n sin\, nx) \equiv \sum_{n=0}^{\infty} A_n(x), \tag{1.8}$$

with n^{th} partial sums $s_n(f; x)$ called trigonometric polynomial of degree (order) n of the first $(n + 1)$ terms of the Fourier series of f.
The conjugate series of Fourier series (1.8) is given by

$$\sum_{n=1}^{\infty}(b_n cos\, nx - a_n sin\, nx) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{1.9}$$

We note that E_n^q is also trigonometric polynomial of degree (or order) n.
We use the following notation throughout this paper.

$$\psi_x(t) = \psi(t) = f(x+t) - f(x-t),$$

$$\widetilde{G}_n(t) = \frac{1}{2\pi(1+q)^n \sin t/2}\left[\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k} \cos\Big(k + \frac{1}{2}\Big)t\right].$$

The conjugate function $\widetilde{f}(x)$ is defined for almost all x by

$$\widetilde{f}(x) = -\frac{1}{2\pi}\int_0^{\pi} \psi(t) \cot\,(t/2) dt = \lim_{h\to 0}\Big(-\frac{1}{2\pi}\int_h^{\pi} \psi(t) \cot(t/2) dt\Big).$$

Furthermore, C denotes an absolute positive constant, not necessarily the same at each occurrence.
We note that the series, conjugate to a Fourier series, is not necessarily a Fourier series.
Hence a separate study of conjugate series is desirable and attracted the attention of researchers.

2 KNOWN RESULTS

Chandra [7] has studied the degree of approximation to a function $f \in Lip\,\alpha$ $(0 < \alpha \leq 1)$ by $(E, q), q > 0$ of Fourier series (1.8) by proving the following theorem:

Theorem 2.1. *The degree of approximation of a periodic function $f(x)$ with period 2π and belonging to the class $Lip\,\alpha$ by Eulers mean of its Fourier series is given by*

$$\max |f(x) - T_n^q(x)| = O\left(n^{-\alpha/2}\right) \tag{2.1}$$

where $T_n^q(x)$ is the n^{th} Euler mean of order $q > 0$ of the sequence $\{s_n\}$ of partial sums of the Fourier series (1.8) of the function f at a point x in $[-\pi, \pi]$.

Shukla [21] improved Theorem 2.1 by extending to a function $f \in Lip(\alpha, r)$ by (E, q) matrix means of the conjugate series (1.9) of its Fourier series (1.8). He proved:

Theorem 2.2. *Let $f \in Lip(\alpha, r), 0 < \alpha \leq 1,\ r \geq 1$ be a 2π-periodic and Lebesgue integrable function of 't' in the interval $[-\pi, \pi]$. If*

$$\left\{\int_0^t u^{(1-\alpha)}|\psi(u)|^r du\right\}^{\frac{1}{r}} = O(t), \tag{2.2}$$

and

$$\left\{\int_1^\pi u^{(-\delta-\alpha)}|\psi(u)|^r du\right\}^{\frac{1}{r}} = O(t^{-\alpha-1}), \tag{2.3}$$

where δ is an arbitrary number such that $s(2 - \delta) > 1$, s being conjugate to $r \geq 1$ with $r^{-1} + s^{-1} = 1$, then the degree of approximation of the conjugate to a function $f \in Lip(\alpha, r)$, by (E, q) means, $q > 0$, of the conjugate series (1.9) of its Fourier series (1.8) will be given by

$$\max \left\|\widetilde{f}(x) - \widetilde{E}_n^q(x)\right\| = O\left\{n^{-\frac{\alpha}{2}+\frac{1}{2r}}\right\}, \tag{2.4}$$

where $\widetilde{E}_n^q(x)$ is n^{th} (E, q) mean of the sequence $\widetilde{s}_n(x)$ of partial sums of the conjugate series (1.9) of the Fourier series (1.8) of the function f at every point x in $[-\pi, \pi]$ at which

$$\widetilde{f}(x) = \frac{1}{2\pi}\int_0^\pi \psi(t)\cos t/2dt \tag{2.5}$$

exists.

Very recently, Mishra et al. [11] has determined the degree of approximation of the function $\widetilde{f}(x)$, conjugate to 2π periodic signal (function) f belonging to $Lip(\xi(t), r)$ class by Euler (E, q), $q > 0$ summability means, which in turn generalizes Theorem 2.1 of Chandra [7] and Theorem 2.2 of Shukla [21]. They proved:

Theorem 2.3. *If $\widetilde{f}(x)$ is conjugate to a 2π- periodic signal (function) f belonging to $Lip(\xi(t), r)$- class, then its degree of approximation by (E, q) means of conjugate series of Fourier series (1.9) is given by*

$$\|\widetilde{E}_n^q(f; x) - \widetilde{f}(x)\|_r = O\left\{(n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right\}, \tag{2.6}$$

provided positive increasing $\xi(t)$ satisfies the following conditions:

$$\left\{\int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O(1), \tag{2.7}$$

$$\left\{\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\psi_x(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O((n+1)^\delta), \tag{2.8}$$

and

$$\{\xi(t)/t\} \text{is non-increasing in ‘}t\text{’}, \tag{2.9}$$

where δ is an arbitrary number such that $s(1 - \delta) - 1 > 0$, $r^{-1} + s^{-1} = 1$for $1 \leq r \leq \infty$, conditions (2.7) and (2.8) hold uniformly in x and $\widetilde{E}_n^q$ is n^{th} (E, q) means of the series (1.9).

3 MAIN RESULTS

The aim of the present paper is to extend the above three theorems i.e. Theorems 2.1, Theorem 2.2 and Theorem 2.3 on the degree of approximation of signal $\widetilde{f}(x)$, conjugate to a 2π-periodic signal f belonging to generalized weighted Lipschitz $W(L^r, \xi(t))(r \geq 1)$ class by Euler $(E, q)(q > 0)$ summability transform with a proper set of conditions. More precisely, we prove:

Theorem 3.1. *Let $f \in W(L^r, \xi(t))(r \geq 1)$ be a 2π-periodic signal (function) and Lebesgue integrable on $[0, 2\pi]$. $\widetilde{f}$ be conjugate of f. Let $0 \leq \beta \leq 1 - \frac{1}{r}$. Then the degree of approximation of $\widetilde{f}$ by Euler (E, q) $(q > 0)$ means of the conjugate Fourier series of (1.9) is given by*

$$\left\| \widetilde{E}_n^q(f; x) - \widetilde{f}(x) \right\| = O\left\{ (\sqrt{n})^{\beta + \frac{1}{r}} \xi\left(\frac{1}{\sqrt{n}} \right) \right\}, \tag{3.1}$$

provided the function ξ (which is positive and monotonic increasing) satisfies the following:

$$(i) \ \left\{ \frac{\xi(t)}{t} \right\} \text{ is non-increasing with regards to } t; t > 0, \tag{3.2}$$

$$(ii) \ \left\{ \int_0^{\frac{\pi}{\sqrt{n}}} \left(\frac{|\psi_x(t)|}{\xi(t)} \right)^r sin^{\beta r}\left(\frac{t}{2} \right) dt \right\}^{1/r} = O(1), \tag{3.3}$$

and

$$(iii) \ \left\{ \int_{\frac{\pi}{\sqrt{n}}}^{\pi} \left(\frac{t^{-\delta} |\psi_x(t)|}{\xi(t)} \right)^r dt \right)^{1/r} = O((\sqrt{n})^{\delta}), \tag{3.4}$$

where δ is such that $(\beta - \delta)\left(1 - \frac{1}{r}\right)^{-1} - 1 > 0$. Further (ii) and (iii) should hold uniformly in x.

Note 3.2 Using condition (3.2), we get $\xi\left(\frac{\pi}{\sqrt{n}}\right) \leq (\pi)\xi\left(\frac{1}{\sqrt{n}}\right)$, for $\left(\frac{\pi}{\sqrt{n}}\right) \geq \left(\frac{1}{\sqrt{n}}\right)$.

Note 3.3 If $\beta = 0$, then our main Theorem 3.1 reduces to Theorem 2.3 and thus generalizes theorem of Mishra et al. [11].

Note 3.4 $\xi\left(\frac{1}{\sqrt{n}}\right) = \left(\frac{1}{\sqrt{n}}\right)^{\alpha}$, then our main Theorem 3.1 reduces to Theorem 2.2, and thus generalizes the theorem of Shukla [21].

Note 3.5 The transform $(E, q)(q > 0)$ plays an important role in Signal Analysis and the theory of Machines in Mechanical Engineering.

4 LEMMA

In order to prove our main theorem, we require the following lemma.

Lemma 4.1. For $0 \leq t \leq \pi$, we have

$$|\widetilde{G}_n(t)| = O\left(t^{-1} e^{-2qnt^2/\{\pi(1+q)\}^2} \right).$$

Proof. We know that

$$\begin{aligned} |\widetilde{G}_n(t)| &= \left| \frac{1}{2\pi(1+q)^n \sin(t/2)} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \cos\left(k + 1/2\right)t \right| \\ &\leq \frac{(1+q)^{-n}}{t} \left| R \sum_{k=0}^{n} \binom{n}{k} q^{n-k} e^{ikt} \right| |e^{it/2}| \\ &= O\left(\frac{1}{t}\right) [|q + e^{it}|^n (1+q)^{-n}] \\ &= O\left(\frac{1}{t}\right) (1+q)^{-n} (1 + q^2 + 2q\cos t)^{n/2} \\ &= O\left(\frac{1}{t}\right) O\left[e^{\frac{-2qnt^2}{(\pi(1+q))^2}} \right], \end{aligned}$$

because

$$\begin{aligned} (1+q)^{-2}(1 + q^2 + 2q\cos t) &= 1 - \frac{4q\sin^2 t/2}{(1+q)^2} \\ &= 1 - \frac{4qt^2}{\pi^2(1+q)^2} \\ &\leq e^{\frac{-4qt^2}{(\pi(1+q))^2}} \end{aligned}$$

and

$$e^x(1-x) < 1, \text{for } 0 < x < 1.$$

So that

$$\begin{aligned} (1+q)^{-n}(1 + q^2 + 2q\cos t)^{n/2} &= [(1+q)^{-2}(1 + q^2 + 2q\cos t)]^{n/2} \\ &\leq e^{\frac{-2qnt^2}{(\pi(1+q))^2}} \end{aligned}$$

This completes the proof of Lemma 4.1. □

5 PROOF OF THEOREM 3.1

Let $\tilde{s}_n(f;x)$ denote the n^{th} partial sum of series (1.9), then we have

$$\widetilde{s}_n(f;x) - \widetilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi} \psi_x(t)\frac{\cos(n+1/2)t}{\sin(t/2)}dt.$$

Denoting (E,q) means of transform $\{\widetilde{s}_n(f;x)\}$ by $\widetilde{E}_n^q(f;x)$, we write

$$\begin{aligned}
\widetilde{E}_n^q(f;x) - \widetilde{f}(x) &= \frac{1}{2\pi(1+q)^n}\int_0^{\pi} \frac{\psi_x(t)}{\sin(t/2)}\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\cos(k+1/2)t\right\}dt \\
&= \int_0^{\pi}\psi_x(t)\widetilde{G}_n(t)\,dt \\
&= \left[\int_0^{\pi/\sqrt{n}} + \int_{\pi/\sqrt{n}}^{\pi}\right]\psi_x(t)\widetilde{G}_n(t)dt \\
&= I_1 + I_2, \text{say.} \qquad (5.1)
\end{aligned}$$

Clearly,
$|\psi_x(x+t) - \psi(t)| \leq |f(u+x+t) - f(u+x)| + |f(u-x-t) - f(u-x)|$
Hence by Minkowski's inequality, we have

$$\begin{aligned}
\left\{\int_0^{2\pi}|\psi_x(x+t) - \psi(t)\sin^{\beta}(x/2)|^r dx\right\}^{1/r} &\leq \left\{\int_0^{2\pi}|(f(u+x+t) - f(u+x))\sin^{\beta}(x/2)|^r dx\right\}^{1/r} + \\
&\quad \left\{\int_0^{2\pi}|(f(u-x-t) - f(u-x))\sin^{\beta}(x/2)|^r dx\right\}^{1/r} \\
&= O(\xi(t)).
\end{aligned}$$

Then $f \in W(L^r, \xi(t)) \implies \psi_x \in W(L^r, \xi(t))$.
Applying Hölder's inequality, condition (3.2) and Lemma 4.1, note 3.2, $(\sin(t/2))^{-1} \leq \frac{\pi}{t}$, for $0 \leq t \leq \pi$ and second mean value theorem for integrals, we get

$$\begin{aligned}
|I_1| &\leq \int_0^{\pi/\sqrt{n}}|\psi_x(t)||\widetilde{G}_n(t)|dt \\
&\leq \left[\int_0^{\pi/\sqrt{n}}\left(\frac{|\psi_x(t)|\sin^{\beta}(t/2)}{\xi(t)}\right)^r dt\right]^{1/r}\left[\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/\sqrt{n}}\left(\frac{\xi(t)|\widetilde{G}_n(t)|}{\sin^{\beta}(t/2)}\right)^s dt\right]^{1/s} \\
&= O(1)\left[\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/\sqrt{n}}\left(\frac{\xi(t)}{t^{1+\beta}}e^{-2snt^2/\{\pi(1+s)\}^2}\right)^s dt\right]^{1/s} \\
&= O\left[\xi\left(\frac{\pi}{\sqrt{n}}\right)\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/\sqrt{n}}t^{-(1+\beta)s}dt\right]^{1/s} \\
&= O\left[(\sqrt{n})^{\beta+1-\frac{1}{s}}\xi\left(\frac{1}{\sqrt{n}}\right)\right] \\
&= O\left[(\sqrt{n})^{\beta+\frac{1}{r}}\xi\left(\frac{1}{\sqrt{n}}\right)\right], r^{-1} + s^{-1} = 1. \qquad (5.2)
\end{aligned}$$

Again using Hölder's inequality, Lemma 4.1, $(\sin\ t/2)^{-1} \leq \pi/t$, for $0 \leq t \leq \pi$, condition (3.4), note 3.2 and second mean value theorem for integrals, we have

$$
\begin{aligned}
|I_2| &\leq \int_{\pi/\sqrt{n}}^{\pi} |\psi_x(t)||\widetilde{G}_n(t)|dt \\
|I_2| &\leq \left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{t^{-\delta}|\psi_x(t)|sin^{\beta}(t/2)}{\xi(t)}\right)^r dt\right]^{1/r}\left[\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)|\widetilde{G}_n(t)|}{t^{-\delta}sin^{\beta}(t/2)}\right)^s dt\right]^{1/s} \\
&= O\left[\int_{\pi/\sqrt{n}}^{\pi}\left|\frac{t^{-\delta}\psi_x(t)}{\xi(t)}\right|^r dt\right]^{1/r}\left[\int_{\pi/\sqrt{n}}^{\pi}\left|\frac{\xi(t)e^{-2snt^2/\{\pi(1+s)\}^2}}{t^{1-\delta+\beta}}\right|^s dt\right]^{1/s} \\
&= O((\sqrt{n})^{\delta})\left\{\int_{\pi/\sqrt{n}}^{\pi}\left(\frac{\xi(t)}{t^{1-\delta+\beta}}\right)^s dt\right\}^{1/s} \\
&= O((\sqrt{n})^{\delta})\left(\int_{1/\sqrt{n}}^{\sqrt{n}/\pi}\left(\frac{\xi(1/y)}{y^{1-\delta+\beta}}\right)^s\frac{dy}{y^2}\right)^{1/s} \\
&= O\left((\sqrt{n})^{\delta}\left(\frac{\sqrt{n}}{\pi}\right)\xi\left(\frac{\pi}{\sqrt{n}}\right)\right)\left(\int_{\epsilon_1}^{\frac{\sqrt{n}}{\pi}} y^{(\beta-\delta)s-2}dy\right)^{1/s}, \text{ for some} \frac{1}{\pi} < \epsilon_1 < \frac{\sqrt{n}}{\pi}, \\
&= O\left((\sqrt{n})^{\delta+1}\xi\left(\frac{1}{\sqrt{n}}\right)\right)\left(\frac{(\sqrt{n})^{(\beta-\delta)s-1}-(\epsilon_1)^{(\beta-\delta)-s-1}}{(\beta-\delta)s-1}\right)^{1/s} \\
&= O\left((\sqrt{n})^{\delta+1}\xi\left(\frac{1}{\sqrt{n}}\right)(\sqrt{n})^{(\beta-\delta-1/s)}\right) \\
&= O\left((\sqrt{n})^{\beta-\delta-1/s}\xi\left(\frac{1}{\sqrt{n}}\right)\right) \\
&= O\left((\sqrt{n})^{\beta+1/r}\xi\left(\frac{1}{\sqrt{n}}\right)\right), \qquad (5.3)
\end{aligned}
$$

in view of increasing nature of $y\xi(1/y),\ r^{-1}+s^{-1}=1$.
On collecting (5.1) to (5.3), we get

$$|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)| = O\left((\sqrt{n})^{\beta+1/r}\xi\left(\frac{1}{\sqrt{n}}\right)\right). \qquad (5.4)$$

Now, using the L_r -norm of a function, we get

$$
\begin{aligned}
\|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)\|_r &= \left\{\frac{1}{2\pi}\int_0^{2\pi}|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)|^r dx\right\}^{1/r} \\
&= O\left(\int_0^{2\pi}\left((\sqrt{n})^{\beta+1/r}\xi\left(\frac{1}{\sqrt{n}}\right)\right)^r dx\right)^{1/r} \\
&= O\left((\sqrt{n})^{\beta+1/r}\xi\left(\frac{1}{\sqrt{n}}\right)\left(\int_0^{2\pi}dx\right)^{1/r}\right) \\
&= O\left((\sqrt{n})^{\beta+1/r}\xi\left(\frac{1}{\sqrt{n}}\right)\right).
\end{aligned}
$$

This completes the proof of Theorem 3.1.

6 COROLLARIES

From the point of view of the applications, sharper estimates of infinite matrices are useful to get bounds for the lattice norms (which occur in solid state physics) of matrix valued functions and enables to investigate perturbations of matrix valued functions and compare them. Some interesting applications can be seen in [22]. The following corollaries may be derived from our main Theorem 3.1.

Corollary 1. *If $\beta = 0$, then the generalized weighted Lipschitz $W(L^r, \xi(t))(r \geq 1)$-class reduces to the class $Lip(\xi(t), r)$ and the degree of approximation of a function $\widetilde{f}(x)$, conjugate to a 2π- periodic function $f \in Lip(\xi(t), r)$, is given by*

$$\left|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)\right| = O\left((\sqrt{n})^{\frac{1}{r}}\xi\left(\frac{1}{\sqrt{n}}\right)\right), \qquad (6.1)$$

Proof. The result follows by setting $\beta = 0$ in (3.2); we have

$$\|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)\|_r = \left\{\frac{1}{2\pi}\int_0^{2\pi}|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)|^r dx\right\}^{1/r} = O\left((\sqrt{n})^{1/r}\xi\left(\frac{1}{(\sqrt{n})}\right)\right),\ r \geq 1.$$

Thus, we get

$$|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)| \le \left\{\frac{1}{2\pi}\int_0^{2\pi}|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)|^r dx\right\}^{1/r} = O\Big((\sqrt{n})^{1/r}\xi\Big(\frac{1}{(\sqrt{n})}\Big)\Big),\ r\ge 1.$$

This completes the proof of Corollary 1. □

Corollary 2. *If $\beta=0$, $\xi(t)=t^\alpha, 0<\alpha\le 1$, then the generalized weighted Lipschitz $W(L^r,\xi(t))(r\ge 1)$-class reduces to the class $Lip(\alpha,r)$, $(1/r)<\alpha<1$ and the degree of approximation of a function $\widetilde{f}(x)$, conjugate to a 2π-periodic function f belonging to the class $Lip(\alpha,r)$, is given by*

$$\left|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)\right| = O\left(\frac{1}{(\sqrt{n})^{\alpha-1/r}}\right).$$

Proof. Putting $\beta=0, \xi(t)=t^\alpha, 0<\alpha\le 1$ in Theorem (3.1), we have

$$\|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)\|_r = \left\{\frac{1}{2\pi}\int_0^{2\pi}|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)|^r dx\right\}^{1/r}$$

or,

$$O\Big((\sqrt{n})^{\beta+1/r}\xi\Big(\frac{1}{(\sqrt{n})}\Big)\Big) = \left\{\frac{1}{2\pi}\int_0^{2\pi}|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)|^r dx\right\}^{1/r}$$

or,

$$O(1) = \left\{\frac{1}{2\pi}\int_0^{2\pi}|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)|^r dx\right\}^{1/r} O\left(\frac{1}{(\sqrt{n})^{\beta+1/r}\xi\Big(\frac{1}{(\sqrt{n})}\Big)}\right),$$

since otherwise the right hand side of the above equation will not be $O(1)$.
Hence

$$|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)| = O\left(\left(\frac{1}{(\sqrt{n})}\right)^\alpha (\sqrt{n})^{1/r}\right) = O\left(\frac{1}{(\sqrt{n})^{\alpha-1/r}}\right)$$

This completes the proof of Corollary 2. □

Corollary 3. *If $\beta=0$, $\xi(t)=t^\alpha, 0<\alpha<1$ and $r\to\infty$ in (3.1), then $f\in Lip\ \alpha$. In this case, the degree of approximation of a function $\widetilde{f}(x)$, conjugate to a 2π- periodic function f belonging to the class Lip $\alpha(0<\alpha<1)$ is given by*

$$\left|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)\right| = O\left((\sqrt{n})^{-\alpha}\right).$$

Proof. For $r\to\infty$ in Corollary 2, we get

$$\|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)\|_\infty = \sup_{0\le x\le 2\pi}|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)| = O((\sqrt{n})^{-\alpha}).$$

Thus, we have

$$\begin{aligned}|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)| &\le \|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)\|_\infty\\ &= \sup_{0\le x\le 2\pi}|\widetilde{E}_n^q(f;x)-\widetilde{f}(x)|\\ &= O((\sqrt{n})^{-\alpha}).\end{aligned}$$

This completes the proof of Corollary 3. □

7 EXAMPLE

In this example, we see how the E_n^1 summability of partial sums of a Fourier series is better behaved than the sequence of partial sums $s_n(x)$ itself.
Let

$$f(x)=\begin{cases}-1, & -\pi\le x<0,\\ 1, & 0\le x<\pi,\end{cases}$$

with $f(x+2\pi)=f(x)$ for all real x. Fourier series of $f(x)$ is given by

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}sin\,n\,x,\ -\pi\le x\le\pi. \tag{7.1}$$

Then n^{th} partial sum $s_n(x)$ of Fourier series (7.1) is given by

$$s_n(x)=\frac{4}{\pi}(sin\,x+(1/3)\,sin\,3\,x+...+(1/n)\,sin\,n\,x), \tag{7.2}$$

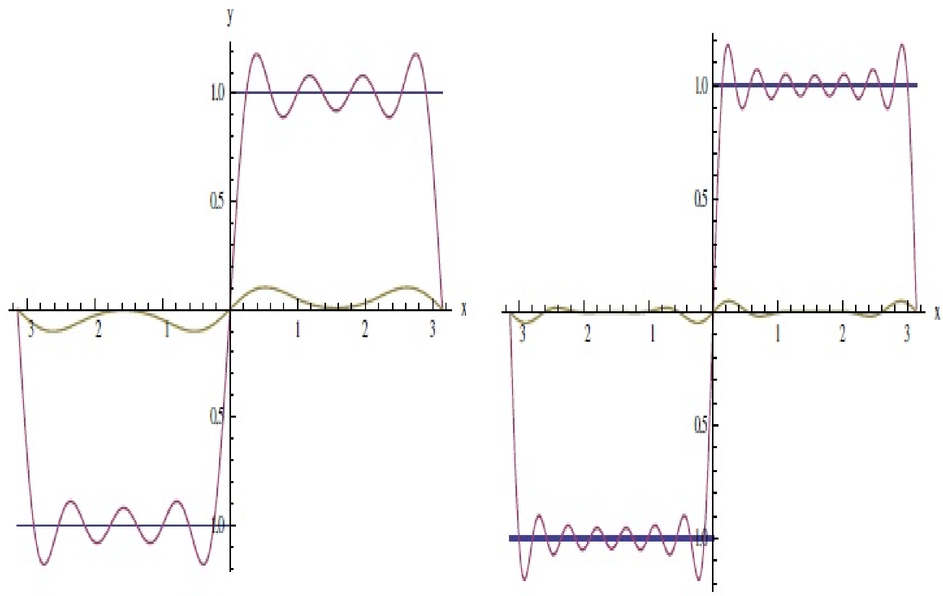

Figure 1: Graph of $f(x)$ (blue), $s_n(x)$ (pink), $E^1_n(f;x)$ (yellow), $n = 7$ and 14.

The E^1_n summability is defined as the n^{th} partial sum of E^1_n summability and we denote it by E^1_n. If

$$E^1_n(f;x) = \frac{1}{(2)^n} \sum_{k=0}^{n} \binom{n}{k} s_k(f;x) \to s, \text{ as } n \to \infty \tag{7.3}$$

In the Figure 1, we observe that $E^1_n(f;x)$ converges to $f(x)$ faster than $s_n(x)$ in the interval $[-\pi, \pi]$. We further note that near the points of discontinuities i.e. $-\pi$, 0 and π , the graph of s_7 and s_{14} show peaks and move closer the line passing through points of discontinuity as n increases (Gibbs phenomenon), but in the graph of $E^1_n(f;x), n = 7, 14$ the peaks become flatter. The Gibbs phenomenon is an overshoot a peculiarity of the Fourier series and other eigen function series at a simple discontinuity i.e. the convergence of Fourier series is very slow at the point of discontinuity. Thus the product summability means of the Fourier series of $f(x)$ overshoot the Gibbs phenomenon and show the smoothing effect of the method. Thus $E^1_n(f;x)$ is the better approximant than $s_n(x)$.

8 CONCLUSION

The results of our lemmas and theorems are more general rather than the results of any other previous proved lemmas and theorems, which will be enrich the literate of Applications of signals in approximation theory and convergence estimates in the theory of approximations by linear operators. The researchers and professionals working or intend to work in areas of analysis and its applications will find this research article to be quite useful. Consequently, the results so established may be found useful in several interesting situation appearing in the literature on Mathematical Analysis, Applied Mathematics and Mathematical Physics etc.

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