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Approximating Solutions of Nonlinear First Order Ordinary Differential Equations

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ABSTRACT: In this paper, the authors prove some algorithms for the existence as well as approximations of the solutions for initial as well as periodic boundary value problems of nonlinear first order ordinary differential equations under generalized monotonicity conditions. We rely our main results on a recent hybrid fixed point theorem of Dhage (2014) in partially ordered normed linear spaces and are obtained under weaker partial continuity and partial compactness type conditions. Our results are also illustrated by some numerical examples.

1 INTRODUCTION

The study of nonlinear differential equations via successive approximations has been a topic of great interest since long time. It is Picard who first devised a constructive method for the initial value problems of nonlinear first order ordinary differential equations in terms of a sequence of successive approximations converging to a unique solution of the related differential equations. The method is commonly known as Picard's iteration method in nonlinear analysis and frequently used for nonlinear equations in the literature. It employs the Lipschitz condition of the nonlinearities together with a certain restriction on Lipschitz constant. The Picard's method is further abstracted to metric spaces by Banach which thereby made it possible to relax the condition on Lipschitz constant. Many attempts have been made in the literature to weaken the Lipschitz condition for the existence of unique solution of nonlinear equations. Nieto and Lopez [11] weakened Lipschitz condition to partial Lipschitz condition guaranteeing the conclusion of the Picard's method under certain additional conditions. But in any circumstances the hypothesis of Lipschitz condition is unavoidable to guarantee the conclusion of Picard's method for nonlinear problems. Very recently, the present author in [3] proved an abstract hybrid fixed point theorem in the setting of a partially ordered metric space without using any kind of geometric condition and still the conclusion of Picard's method holds. However, in this case the order relation and the metric are required to satisfy certain compatibility condition. This method is commonly known as *Dhage iteration method* in the literature and applied to several nonlinear differential integral equations. In this paper, we use this iteration method based on a hybrid fixed point theorem in the study of initial and boundary value problems of nonlinear first order ordinary differential equations under generalized monotonic conditions and derive a stronger conclusion than that of Picard method.

The rest of the paper will be organized as follows. In Section 2 we give some preliminaries and a key fixed point theorem that will be used in subsequent part of the paper. In Section 3 we discuss the existence result for initial value problems and in Section 4 we discuss the existence result for periodic boundary value problems of first order ordinary differential equations.

2 AUXILIARY RESULTS

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers is preserved by $\preceq$. A few details of such spaces appear in Dhage [2] and the references therein.

Two elements x and y in E are said to be **comparable** if either the relation $x \preceq$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [10], Nieto and Lopez [11] and the references therein.

We need the following definitions in the sequel.

Definition 2.1. *A mapping $\mathcal{T} : E \to E$ is called* **isotone** *or* **monotone nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called* **monotone nonincreasing** *if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called* **monotonic** *or simply* **monotone** *if it is either monotone nondecreasing or monotone nonincreasing on E.*

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The following terminologies may be found in any book on nonlinear analysis and applications. See Granas and Dugundji [9] Heikkila and Lakshmikantham [10] and the references therein.

An operator $\mathcal{T}$ on a normed linear space E into itself is called **compact** if $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called **totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a relatively compact subset of E. If $\mathcal{T}$ is continuous and totally bounded, then it is called **completely continuous** on E.

Definition 2.2 (Dhage [3]). *A mapping $\mathcal{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.3 (Dhage [2, 3]). *An operator $\mathcal{T}$ on a partially normed linear space E into itself $\mathcal{T}$ is called* **partially bounded** *if $T(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called* **uniformly partially bounded** *if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant. $\mathcal{T}$ is called* **partially compact** *if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called* **uniformly partially compact** *if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called* **partially partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Remark 2.4. Note that every compact mapping on a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.

Definition 2.5 (Dhage [2]). *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* **compatible** *if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called* **Janhavi** *if the order relation and the metric or norm are compatible in it.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property. In general every finite dimensional Banach space with a standard norm and an order relation is a Janhavi Banach space.

The following fixed point result is a slight improvement of the applicable hybrid fixed point theorem proved in Dhage [2] in a partially ordered metric space.

Theorem 2.6 (Dhage [3]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain C of E. Let $\mathcal{T} : E \to E$ be a partially continuous, nondecreasing and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the operator equation $\mathcal{T}x = x$ has a solution x^* in E and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Remark 2.7. The regularity of E in above Theorem 2.6 may be replaced with a stronger continuity condition of the operator $\mathcal{T}$ on E which is a result proved in Dhage [2].

3 INITIAL VALUE PROBLEMS

Given a closed and bounded interval $J = [0, T]$ of the real line $\mathbb{R}$ for some $T > 0$, consider the initial value problem (in short IVP) of first order ordinary nonlinear hybrid differential equation,

$$\left.\begin{aligned} x'(t) &= f(t, x(t)), \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1}$$

for all $t \in J$, where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

By a *solution* of the IVP (1) we mean a function $x \in C^1(J, \mathbb{R})$ that satisfies equation (1), where $C^1(J, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J.

The IVP (1) is well-known in the literature and discussed at length for existence as well as other aspects of the solutions. In the present paper and it is proved that the existence of the solutions may be proved under weaker partially continuity and partially compactness type conditions.

The equivalent integral formulation of the IVP (1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{2}$$

and

$$x \le y \iff x(t) \le y(t) \tag{3}$$

for all $t \in J$. Clearly, $C(J,\mathbb{R})$ is a partially ordered Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\le$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ is regular and has some nice properties w.r.t. the above order relation $\le$ in it. The following lemma follows by an application of Arzellá-Ascolli theorem.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}), \le, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\le$ defined by (2) and (3) respectively. Then $\|\cdot\|$ and $\le$ are compatible in every partially compact subset of $C(J,\mathbb{R})$.*

Proof. The proof of the lemma appears in Dhage [5]. Since it is not well-known, we give the details of it. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \le x_2(t) \le \cdots \le x_n(t) \le \cdots \tag{ND}$$

for each $t \in J$.

Suppose that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}$ of the monotone real sequence $\{x_n(t)\}$ is convergent. By monotone characterization, the original sequence $\{x_n(t)\}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in J$. This shows that the sequence $\{x_n\}$ converges point-wise to a point $x \in S$. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}$ is an equicontinuous sequence by Arzelá-Ascolli theorem. Hence $\{x_n\}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\le$ are compatible in S. This completes the proof. □

We need the following definition in what follows.

Definition 3.2. *A function $u \in C^1(J,\mathbb{R})$ is said to be a lower solution of the IVP (1) if it satisfies*

$$\left.\begin{aligned} u'(t) &\le f(t,u(t)), \\ u(0) &\le x_0, \end{aligned}\right\} \tag{$**$}$$

for all $t \in J$. Similarly, an upper solution $v \in C^1(J,\mathbb{R})$ to the IVP (1) is defined on J.

We consider the following set of assumptions in what follows:

(H_1) There exists a real number $\lambda > 0$ such that the map $x \mapsto f(t,x) + \lambda x$ is monotone nondecreasing for each $t \in J$.

(H_2) The IVP (1) has a lower solution $u \in C^1(J,\mathbb{R})$.

Consider the IVP

$$\left.\begin{aligned} x'(t) + \lambda x(t) = \tilde{f}(t,x(t)),\ t \in J, \\ x(0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{4}$$

where $\tilde{f} : J \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\tilde{f}(t,x) = f(t,x) + \lambda x. \tag{5}$$

Remark 3.3. A function $u \in C^1(J,\mathbb{R})$ is a lower solution of the IVP (1) if and only if it is a solution of the IVP (4) defined on J. A Similar assertion is also true for an upper solution. A function $u \in C(J,\mathbb{R})$ is a solution of the IVP (1) if and only if it is a lower as well as an upper solution of the same defined on J.

Consider the following assumption in what follows.

(H_3) There exists a constant $K > 0$ such that $|\tilde{f}(t,x)| \le K$ for all $t \in J$ and $x \in \mathbb{R}$.

Lemma 3.4. *Assume that hypothesis (H_3) holds. Then a function $u \in C(J,\mathbb{R})$ is a solution of the IVP (1) if and only if it is a solution of the nonlinear integral equation,*

$$x(t) = x_0 e^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda s} \tilde{f}(s,x(s))\,ds \tag{6}$$

for all $t \in J$.

Theorem 3.5. *Assume that hypotheses (H_1) through (H_3) hold. Then the IVP (1) has a solution x^* defined on J and the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by*

$$x_{n+1}(t) = x_0 e^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda s} \tilde{f}(s,x_n(s))\,ds,\ t \in J, \tag{7}$$

where $x_1(t) = u(t)$ $t \in J$, converges monotonically to x^*.

Proof. Set $E = C(J, \mathbb{R})$. Then, in view of Lemma 3.1, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define the operator $\mathcal{T}$ on E by

$$\mathcal{T}x(t) = x_0 e^{-\lambda t} + e^{-\lambda t}\int_0^t e^{\lambda s}\tilde{f}(s, x(s))\,ds, \ \ t \in J. \tag{8}$$

From the continuity of the integrals, it follows that $\mathcal{T}$ defines the maps $\mathcal{T} : E \to E$. Now by Lemma 3.4, the IVP (1) is equivalent to the operator equation

$$\mathcal{T}x(t) = x(t), \ \ t \in J. \tag{9}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.6. This is achieved in the series of following steps.

Step I: *$\mathcal{T}$ is a nondecreasing operator on E.*

Let $x, y \in E$ be such that $x \geq y$. Then by hypothesis (H_1), we obtain

$$\begin{aligned}
\mathcal{T}x(t) &= x_0 e^{-\lambda t} + e^{-\lambda t}\int_{t_0}^t e^{\lambda s}\tilde{f}(s, x(s))\,ds \\
&\geq x_0 e^{-\lambda t} + e^{-\lambda t}\int_{t_0}^t e^{\lambda s}\tilde{f}(s, y(s))\,ds \\
&= \mathcal{T}y(t),
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}$ is a nondecreasing operator on E into E.

Step II: *$\mathcal{T}$ is a partially continuous operator on E.*

Let $\{x_n\}$ be a sequence in a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty}\left[x_0 e^{-\lambda t} + e^{-\lambda t}\int_0^t e^{\lambda s}\tilde{f}(s, x_n(s))\,ds\right] \\
&= x_0 e^{-\lambda t} + e^{-\lambda t}\int_0^t e^{\lambda s}\left[\lim_{n\to\infty}\tilde{f}(s, x_n(s))\right]ds \\
&= x_0 e^{-\lambda t} + e^{-\lambda t}\int_0^t e^{\lambda s}\tilde{f}(s, x(s))\,ds \\
&= \mathcal{T}x(t),
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}x_n$ converges to $\mathcal{T}x$ pointwise on J.

Next, we will show that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}
|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| &= \left|e^{-\lambda t_2}\int_0^{t_2} e^{\lambda s}\tilde{f}(s, x_n(s))ds - e^{-\lambda t_1}\int_0^{t_1} e^{\lambda s}\tilde{f}(s, x_n(s))ds\right| \\
&\leq \left|\left(e^{-\lambda t_2} - e^{-\lambda t_1}\right)\int_0^{t_1} e^{\lambda s}\tilde{f}(s, x_n(s))ds\right| \\
&\quad + \left|e^{-\lambda t_2}\int_{t_1}^{t_2} e^{\lambda s}\tilde{f}(s, x_n(s))ds\right| \\
&\leq \left|\left(e^{-\lambda t_2} - e^{-\lambda t_1}\right)\int_0^{T} e^{\lambda s}K\,ds\right| \\
&\quad + \left|\int_{t_1}^{t_2} e^{\lambda s}K\,ds\right| \\
&\to 0 \ \ as \ \ t_1 \to t_2
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{T}x_n \to \mathcal{T}x$ is uniformly and hence $\mathcal{T}$ is a partially continuous operator on E into itself.

Step III: *$\mathcal{T}$ is a partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{T}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{T}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned}
|\mathcal{T}x(t)| &\leq \left|x_0 e^{-\lambda t}\right| + \left|e^{-\lambda t}\int_0^t e^{\lambda s}\tilde{f}(s, x(s))\,ds\right| \\
&\leq |x_0| + \int_0^t e^{\lambda s}|\tilde{f}(s, x(s))|\,ds
\end{aligned}$$

$$\begin{aligned}&\leq |x_0| + \int_0^T e^{\lambda T} K\, ds\\&\leq |x_0| + e^{\lambda T} KT = r,\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{T}x\| \leq r$ for all $x \in C$. Hence $\mathcal{T}$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{T}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| &= \left| e^{-\lambda t_2} \int_0^{t_2} e^{\lambda s} \tilde{f}(s, x(s))\, ds - e^{-\lambda t_1} \int_0^{t_1} e^{\lambda s} \tilde{f}(s, x(s)) ds \right|\\&\leq \left| \left(e^{-\lambda t_2} - e^{-\lambda t_1}\right) \int_0^{t_1} e^{\lambda s} \tilde{f}(s, x(s)) ds \right| + \left| e^{-\lambda t_2} \int_{t_1}^{t_2} e^{\lambda s} \tilde{f}(s, x(s)) ds \right|\\&\leq \left| \left(e^{-\lambda t_2} - e^{-\lambda t_1}\right) \int_0^{T} e^{\lambda s} K\, ds \right|\\&\quad + \left| \int_{t_1}^{t_2} e^{\lambda s} K\, ds \right|\\&\to 0 \quad as \quad t_1 \to t_2\end{aligned}$$

uniformly for all $x \in C$. Hence $\mathcal{T}(C)$ is compact subset of E and consequently $\mathcal{T}$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality $u \leq \mathcal{T}u$.*

By hypothesis (H_2), the IVP (1) has a lower solution u. Then we have

$$\left.\begin{aligned}u'(t) &\leq f(t, u(t))),\\u(0) &\leq x_0,\end{aligned}\right\} \tag{10}$$

for all $t \in J$. Adding $\lambda u(t)$ on both sides of the first inequality in (10), we obtain

$$u'(t) + \lambda u(t) \leq f(t, u(t)) + \lambda u(t), \quad t \in J. \tag{11}$$

Again, multiplying the above inequality (11) by $e^{\lambda t}$,

$$\left(e^{\lambda t} u(t)\right)' \leq e^{\lambda t} \tilde{f}(t, u(t)). \tag{12}$$

A direct integration of (12) from 0 to t yields

$$u(t) \leq x_0 e^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda s} \tilde{f}(s, u(s))\, ds, \tag{13}$$

for all $t \in J$. From definition of the operator $\mathcal{T}$ it follows that

$$u(t) \leq \mathcal{T}u(t),$$

for all $t \in J$. Hence $u \leq \mathcal{T}u$.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.6 and we apply it to conclude that the operator equation $\mathcal{T}x = x$ has a solution. Consequently the integral equation and the IVP (1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (1) converges monotonically to x^*. This completes the proof. □

Remark 3.6. The conclusion of Theorem 3.5 also remains true if we replace the hypothesis (H_2) with the following one:

(H_2') The IVP (1) has an upper solution $v \in C^1(J, \mathbb{R})$.

Example 3.7. *Given a closed and bounded interval $J = [0, 1]$, consider the IVP,*

$$\left.\begin{aligned}x'(t) = \tan^{-1} x(t) - x(t),\\x(0) = 1,\end{aligned}\right\} \tag{14}$$

for all $t \in J$.

Here, $f(t, x) = \tan^{-1} x - x$. Clearly, the functions f is continuous on $J \times \mathbb{R}$. The function f satisfies the hypothesis (H_1) with $\lambda = 1$. Moreover, the function $\tilde{f}(t, x) = \tan^{-1} x$ is bounded on $J \times \mathbb{R}$ with bound $K = \dfrac{\pi}{2}$ and so the hypothesis (H_3) is satisfied.

Since $-2 < \tan^{-1} x < 2$ for all $x \in \mathbb{R}$, any function $u \in C^1(J, \mathbb{R})$ satisfying the linear differential equation

$$\left.\begin{aligned}x'(t) + x(t) = -2,\\x(0) = 1,\end{aligned}\right\} \tag{15}$$

is a lower solution of the IVP (14) on J. Because, in this case, we obtain

$$\left.\begin{aligned} u'(t)+u(t) &\le \tan^{-1} u(t), \\ u(0) &= 1, \end{aligned}\right\} \tag{16}$$

for all $t \in J$. Therefore, solving (15) for unknown function u, we get

$$u(t) = 3e^{-t} - 2, \;\; t \in J. \tag{17}$$

Similarly, any function $v \in C^1(J, \mathbb{R})$ satisfying the linear differential equation

$$\left.\begin{aligned} x'(t)+x(t) &= -2, \\ x(0) &= 1, \end{aligned}\right\} \tag{18}$$

is an upper solution of the IVP (14) on J. Solving the differential equation (18) for the unknown function v yields that

$$v(t) = 2 - e^{-t}, \;\; t \in J. \tag{19}$$

Hence, we apply Theorem 3.5 and conclude that the IVP (14) has a solution x^* defined on J and the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by

$$x_{n+1}(t) = e^{-t} + e^{-t}\int_0^t e^s \tan^{-1} x_n(s)\, ds \tag{20}$$

for all $t \in J$, where $x_1(t) = 3e^{-t} - 2, \; t \in J$, converges monotonically to x^*.

Remark 3.8. In view of Remark 3.6, the existence of solutions x^* of the IVP (14) may be obtained under the assumption of existence of the upper solution v defined on J. Here also we conclude that the IVP (14) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by (20) with $x_1 = 2 - e^{-t}$, converges monotonically to x^*.

4 PERIODIC BOUNDARY VALUE PROBLEMS

Given a closed and bounded interval $J = [0, T]$ of the real line $\mathbb{R}$ for some $T > 0$, consider the periodic boundary value problem (in short PBVP) of first order ordinary nonlinear hybrid differential equation

$$\left.\begin{aligned} x'(t) &= f(t, x(t)), \\ x(0) &= x(T), \end{aligned}\right\} \tag{21}$$

for all $t \in J$, where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

By a *solution* of the PBVP (21) we mean a function $x \in C^1(J, \mathbb{R})$ that satisfies the equation (21), where $C^1(J, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J.

The PBVP (21) is well-known in the literature and discussed at length for existence as well as other aspects of the solutions. In the present paper and it is proved that the existence as well as algorithm of the solutions may be proved for periodic boundary value problems of nonlinear first order ordinary differential equations under weaker partial continuity and partial compactness type conditions.

We need the following definition in what follows.

Definition 4.1. *A function $u \in C^1(J, \mathbb{R})$ is said to be a lower solution of the of PBVP (21) if it satisfies*

$$\left.\begin{aligned} u'(t) &\le f(t, u(t)), \\ u(0) &\le u(T), \end{aligned}\right\} \tag{$**$}$$

for all $t \in J$. Similarly, an upper solution $v \in C^1(J, \mathbb{R})$ to the PBVP (21) is defined on J.

(H_4) The PBVP (21) has a lower solution $u \in C^1(J, \mathbb{R})$.

Let $\lambda \in \mathbb{R}$ be such that $\lambda > 0$ and consider the PBVP

$$\left.\begin{aligned} x'(t) + \lambda x(t) &= \tilde{f}(t, x(t)), \\ x(0) &= x(T), \end{aligned}\right\} \tag{22}$$

for all $t \in J$, where $\tilde{f} : J \times \mathbb{R} \to \mathbb{R}$ and

$$\tilde{f}(t, x) = f(t, x) + \lambda x. \tag{23}$$

Remark 4.2. A function $u \in C^1(J, \mathbb{R})$ is a lower solution of the PBVVP (21) if and only if it is a solution of the PBVP (22) defined on J. A Similar assertion is also true for an upper solution. A function $u \in C^1(J, \mathbb{R})$ is a solution of the PBVVP (21) if and only if it is a lower as well as an upper solution of the same defined on J.

The following useful lemma is obvious and may be found in Nieto and Lopez [12].

Lemma 4.3. *For any $h \in L^1(J, \mathbb{R}^+)$ and $\sigma \in L^1(J, \mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} x'(t) + h(t)x(t) = \sigma(t), \ \ t \in J, \\ x(0) = x(T), \end{aligned}\right\} \tag{24}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_h(t,s)\sigma(s)\,ds \tag{25}$$

where,

$$G_h(t,s) = \begin{cases} \dfrac{e^{H(s)-H(t)+H(T)}}{e^{H(T)}-1}, & 0 \le s \le t \le T, \\ \\ \dfrac{e^{H(s)-H(t)}}{e^{H(T)}-1}, & 0 \le t < s \le T, \end{cases} \tag{26}$$

and $H(t) = \displaystyle\int_0^t h(s)\,ds$.

Notice that the Green's function G_h is continuous and nonnegative on $J \times J$ and therefore, the number

$$M_h := \max\{\, |G_h(t,s)| \,:\, t,s \in [0,T]\,\}$$

exists for all $h \in L^1(J, \mathbb{R}^+)$. In particular, if $h = 1$, then for the sake of convenience we write $G_1(t,s) = G(t,s)$ and $M_1 = M$.

An application of above Lemma 4.3 we obtain

Lemma 4.4. *Suppose that hypothesis (H_2) holds. Then a function $u \in C(J, \mathbb{R})$ is a solution of the PBVP* (21) *if and only if it is a solution of the nonlinear integral equation,*

$$x(t) = \int_0^T G(t,s)\tilde{f}(s, x(s))\,ds \tag{27}$$

for all $t \in J$, where

$$G(t,s) = \begin{cases} \dfrac{e^{\lambda s - \lambda t + \lambda T}}{e^{\lambda T}-1}, & 0 \le s \le t \le T, \\ \\ \dfrac{e^{\lambda s - \lambda t}}{e^{\lambda T}-1}, & 0 \le t < s \le T. \end{cases} \tag{28}$$

Theorem 4.5. *Assume that hypotheses (H_1) and (H_3)-(H_4) hold. Then the PBVP (21) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_{n+1}(t) = \int_0^T G(t,s)\tilde{f}(s, x_n(s))\,ds \tag{29}$$

for all $t \in J$, where $x_0 = u$ converges monotonically to x^*.

Proof. Set $E = C(J, \mathbb{R})$. Then, in view of Lemma 3.1, the norm $\|\cdot\|$ and the order relation $\le$ are compatible in every compact chain C of E.

Define the operator $\mathcal{T}$ on E by

$$\mathcal{T}x(t) = \int_0^T G(t,s)\tilde{f}(s, x(s))\,ds, \ \ t \in J. \tag{30}$$

From the continuity of the integrals, it follows that $\mathcal{T}$ defines the map $\mathcal{T} : E \to E$. Now by Lemma 4.4, the PBVP (21) is equivalent to the operator equation

$$\mathcal{T}x(t) = x(t), \ \ t \in J. \tag{31}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.6. This is achieved in the series of following steps.

Step I: *$\mathcal{T}$ is a monotone nondecreasing operator on E.*

Let $x, y \in E$ be such that $x \ge y$. Then by hypothesis (A_1), we obtain

$$\begin{aligned} \mathcal{T}x(t) &= \int_0^T G(t,s)\tilde{f}(s, x(s))\,ds \\ &\ge \int_0^T G(t,s)\tilde{f}(s, y(s))\,ds \\ &= \mathcal{T}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}$ is a nondecreasing operator on E into E.

Step II: *$\mathcal{T}$ is a partially continuous operator on E.*

Let $\{x_n\}$ be a sequence in a chain C of E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty} \int_0^T G(t,s)\tilde{f}(s,x_n(s))\,ds \\
&= \int_0^T G(t,s)\Big[\lim_{n\to\infty} \tilde{f}(s,x_n(s))\Big]\,ds \\
&= \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds \\
&= \mathcal{T}x(t),
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}x_n$ converges to $\mathcal{T}x$ pointwise on J.

Next, we will show that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary elements. Then,

$$\begin{aligned}
|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| &= \left|\int_0^T G(t_1,s)\tilde{f}(s,x_n(s))\,ds - \int_0^T G(t_2,s)g(s,x_n(s))\,ds\right| \\
&= \left|\int_0^T |G(t_1,s) - G(t_2,s)||\tilde{f}(s,x_n(s))|\,ds\right| \\
&\leq K\int_0^T |G(t_1,s) - G(t_2,s)|\,ds \\
&\to 0 \quad as \quad t_2 - t_1 \to 0
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{T}x_n \to \mathcal{T}x$ is uniformly and hence $\mathcal{T}$ is a partially continuous operator on E.

Step III: *$\mathcal{T}$ is a partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{T}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{T}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned}
|\mathcal{T}x(t)| &= \left|\int_0^T G(t,s)\tilde{f}(s,x(s))\,ds\right| \\
&\leq \int_0^T G(t,s)|\tilde{f}(s,x(s))|\,ds \\
&\leq \int_0^T MK\,ds \\
&\leq MKT = r,
\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{T}x\| \leq r$ for all $x \in C$. Hence $\mathcal{T}(C)$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{T}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned}
|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| &= \left|\int_0^T [G(t_1,s) - G(t_2,s)]\tilde{f}(s,x(s))\,ds\right| \\
&= \int_0^T |G(t_1,s) - G(t_2,s)||\tilde{f}(s,x(s))|\,ds \\
&\leq \int_0^T |G(t_1,s) - G(t_2,s)|\,K\,ds \\
&\to 0 \quad as \quad t_1 \to t_2
\end{aligned}$$

uniformly for all $x \in C$. Hence $\mathcal{T}(C)$ is a compact subset of E and consequently $\mathcal{T}$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality $u \leq \mathcal{T}u$.*

By hypothesis (H_4), the PBVP (21) has a lower solution u. Then we have

$$\left.\begin{aligned}
u'(t) &\leq f(t,u(t)), \\
u(0) &\leq u(T),
\end{aligned}\right\} \tag{32}$$

for all $t \in J$. Adding $\lambda u(t)$ on both sides of the first inequality in (11), we obtain

$$u'(t) + \lambda u(t) \leq f(t, u(t)) + \lambda u(t), \ \ t \in J.$$

Again, multiplying the above inequality by $e^{\lambda t}$,

$$\left(e^{\lambda t} u(t)\right)' \leq e^{\lambda t} \tilde{f}(t, u(t)).$$

A direct integration of above inequality from 0 to t yields

$$e^{\lambda t} u(t) \leq u(0) + \int_0^t e^{\lambda s} \tilde{f}(s, u(s))\, ds, \tag{33}$$

for all $t \in J$. Therefore, in particular,

$$e^{\lambda T} u(T) \leq u(0) + \int_0^T e^{\lambda s} \tilde{f}(s, u(s))\, ds. \tag{34}$$

Now $u(0) \leq u(T)$, so one has

$$u(0) e^{\lambda T} \leq u(T) e^{\lambda T}. \tag{35}$$

From (33) and 35) it follows that

$$e^{\lambda T} u(0) \leq u(0) + \int_0^T e^{\lambda s} \tilde{f}(s, u(s))\, ds \tag{36}$$

which further yields

$$u(0) \leq \int_0^T \frac{e^{\lambda s}}{(e^{\lambda T} - 1)} \tilde{f}(s, u(s))\, ds \tag{37}$$

Substituting (37) in (33) we obtain

$$u(t) \leq \int_0^T G(t, s) \tilde{f}(s, u(s))\, ds$$

From definitions of the operator $\mathcal{T}$ it follows that

$$u(t) \leq \mathcal{T} u(t),$$

for all $t \in J$. Hence $u \leq \mathcal{T} u$.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.6 with $x_0 = u$ and we apply it to conclude that the operator equation $\mathcal{T} x = x$ has a solution. Consequently the integral equation and the PBVP (21) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (29) converges monotonically to x^*. This completes the proof. □

Remark 4.6. The conclusion of Theorem 4.5 also remains true if we replace the hypothesis (H_4) with the following one:

(H_4') The PBVP (21) has an upper solution $v \in C^1(J, \mathbb{R})$.

Example 4.7. *Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the PBVP,*

$$\left.\begin{aligned} x'(t) &= f(t, x(t)) - x(t), \\ x(0) &= x(1), \end{aligned}\right\} \tag{38}$$

for all $t \in J$, where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function defined by

$$f(t, x) = \begin{cases} 1, & \text{if } x \leq 0, \\ 1 + \dfrac{x}{1 + x}, & \text{if } x > 0. \end{cases}$$

Here, $\tilde{f}(t, x) = f(t, x)$. Clearly, the function f is continuous on $J \times \mathbb{R}$. Also f satisfies the hypothesis (H_1) with $\lambda = 1$.

Again, since $\tilde{f}$ is bounded on $J \times \mathbb{R}$ by 2, the hypothesis (H_3) holds. Since $1 \leq f(t, x) < 2$ for all $x \in \mathbb{R}$, any function $u \in C^1(J, \mathbb{R})$ satisfying the linear differential equation

$$\left.\begin{aligned} x'(t) + x(t) &= 1, \\ x(0) &= x(1), \end{aligned}\right\} \tag{39}$$

is a lower solution of the PBVP (38) on J. Because, in this case, we obtain

$$\left.\begin{aligned} u'(t) + u(t) &\leq f(t, u(t)), \\ u(0) &= u(1), \end{aligned}\right\} \tag{40}$$

for all $t \in J$. Therefore, solving (39) for the unknown function u, we get

$$u(t) = \int_0^1 G(t,s)\,ds = 1, \ t \in J. \tag{41}$$

Similarly, any function $v \in C^1(J, \mathbb{R})$ satisfying the linear differential equation

$$\left.\begin{aligned} x'(t) + x(t) = 2, \\ x(0) = x(1), \end{aligned}\right\} \tag{42}$$

is an upper solution of the IVP (38) on J. Solving the differential equation (42) for the unknown function v yields that

$$u(t) = 2\int_0^1 G(t,s)\,ds = 2, \ t \in J. \tag{43}$$

Thus all the hypotheses of Theorem 4.5 are satisfied. Hence we apply Theorem 4.5 and conclude that the of PBVP (38) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by

$$x_{n+1}(t) = \int_0^1 G(t,s) f(s, x_n(s))\,ds, \tag{44}$$

for all $t \in J$, where $x_0 = 1$, converges monotonically to x^*.

Remark 4.8. In view of Remark 4.6, the existence of the solutions x^* of the PBVP (38) may be obtained under the assumption of an upper solution $v(t) = 2$ defined on J and the sequence $\{x_n\}$ defined by (44) with $x_0 = 2$, converges monotonically to x^*.

5 CONCLUSION

From the foregoing discussion it is clear that unlike Schauder fixed point principle, the proofs of Theorems 3.5 and 4.5 do not invoke the construction of a non-empty, closed, convex and bounded subset of the Banach space of navigation which is mapped into itself by the operators related to the given differential equations. The convexity hypothesis is altogether omitted from the discussion and still we have proved the existence of the solutions for the differential equations considered in this paper with stronger conclusion. Similarly, unlike the use of Banach fixed point theorem, Theorems 3.5 and 4.5 do not make any use of any type of Lipschitz condition on the nonlinearity involved in the differential equations (1) and (21), but even then the algorithms for the solutions of the differential equations (1) and (21) are proved in terms of the Picard's iteration scheme. The nature of the convergence of the algorithms is not geometrical and so we are not able to obtain the rate of convergence of the algorithms to the solutions of the related problems. However, in a way we have been able to prove the existence results for the IVP (1) and PBVP (21) under much weaker conditions with a stronger conclusion of the monotone convergence of successive approximations to the solutions than those proved in the existing literature on nonlinear differential equations.

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