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Asymptotic Stability of Comparable Solutions for Nonlinear Quadratic Fractional Integral Equations

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ABSTRACT: In this paper the authors prove a local asymptotic attractivity and stability result for a hybrid nonlinear fractional integral equations under the mixed weaker partially Lipschitz and compactness type conditions. It is shown that the comparable solutions of the considered hybrid nonlinear fractional integral equation are uniformly locally ultimately attractive and asymptotically stable on unbounded intervals of the real line. We base our theory on a recent measure theoretic fixed point theorem of Dhage (2014) in partially ordered spaces and claim that our result is new to the literature.

1 INTRODUCTION

In this paper we present the qualitative analysis of the following nonlinear quadratic fractional integral equation (in short QFIE),

$$x(t) = q(t) + \big[f(t,x(t))\big]\left(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\; g(s,x(s))\,ds\right) \tag{1.1}$$

for all $t \in \mathbb{R}_+ = [0,\infty)$, where $q : \mathbb{R}_+ \to \mathbb{R}$, $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous, Γ is the Euler's gamma function and $1 \leq \alpha < 2$.

By a *solution* of the QFIE (1.1) we mean a function $x \in C(\mathbb{R}_+, \mathbb{R})$ that satisfies the equation (1.1), where $C(\mathbb{R}_+, \mathbb{R})$ is the space of continuous real-valued functions on $\mathbb{R}_+$.

The QFIE (1.1) is general and includes several nonlinear quadratic integral studied earlier as special cases. For example, if $\alpha = 1$, and $J = [0,T] \subset \mathbb{R}_+$, then it reduces to the quadratic integral

$$x(t) = q(t) + \big[f(t,x(t))\big]\left(\int_0^t k(t,s)\,g(s,x(s))\,ds\right),\; t \in J, \tag{1.2}$$

which is discussed in Zhu *et.al.* [15] for monotonicity of the solutions on the bounded interval J of $\mathbb{R}$. The QFIE (1.2) again includes a few others integral equations as special cases. See Zhu *et.al* [15] and the references therein. In this paper, we discuss some local existence results for the above QFIE (1.1) and show that solutions are locally attractive in the long period of time t. Our analysis rely on a measure theoretic fixed point theorem of Dhage (2013) in partially ordered Banach space and it is shown that the sequence of successive approximations constructed in a certain way converges to the solution of QFIE (1.1) under certain mixed Lipschitz and compactness type conditions on the nonlinearities involved in it. The measure of noncompactness used in this paper allows us not only to obtain the existence of solutions of the mentioned functional integral equation but also to characterize the solutions in terms of uniform local ultimate attractivity. This assertion means that all possible comparable solutions of the nonlinear fractional integral equation in question are locally uniformly attractive in the sense of notion defined in the following section.

2 AUXILIARY RESULTS

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. We frequently need the concept of regularity of E in what follows. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guarantying the regularity of the partially ordered normed linear space E may be found in Carl and Heikkilä [4] and the references therein. Again, two elements x and y in E are said to be **comparable** if either the relation $x \preceq$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. The following definitions have been introduced in Dhage [5] and are frequently used in the subsequent part of this paper.

A subset S of E is called **partially bounded** if every chain C in S is bounded. Again S is called a **uniformly partially bounded** if all chains in S are bounded with a unique constant.

Note that every bounded subset of a partially ordered normed linear space is uniformly partially bounded and uniformly partially bounded set in E is partially bounded, but the converse implications may hold.

Definition 2.1. *A mapping $\mathcal{T} : E \to E$ is called* **isotonic** *or* **monotonic** *if it is either a monotone nondecreasing or monotone non-increasing, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ or $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$.*

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Definition 2.2 (Dhage [6]). *A mapping $\mathcal{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is a partially continuous on E, then it is continuous on every chain C contained in E. T is called* **partially bounded** *if $\mathcal{T}(C)$ is a bounded subset of E for all totally ordered sets or chains C in E.*

If C is a chain in E, then C' denotes the set of all limit points of C in E. The symbol $\overline{C}$ stands for the closure of C in E defined by $\overline{C} = C \cup C'$. Thu set $\overline{C}$ is called a closed chain in E. Thus, $\overline{C}$ is the intersection of all closed chains containing C. Clearly, $\inf C, \sup C \in \overline{C}$ provided $\inf C$ and $\sup C$ exist. The $\sup C$ is an element $z \in E$ such that for every $\epsilon > 0$ there exists a $c \in C$ such that $d(c, z) < \epsilon$ and $x \leq z$ for all $x \in C$. Similarly, $\inf C$ is defined in the same way.

In what follows, we denote by $\mathcal{P}_{cl}(E), \mathcal{P}_{bd}(E), \mathcal{P}_{rcp}(E), \mathcal{P}_{ch}(E), \mathcal{P}_{bd,ch}(E), \mathcal{P}_{rcp,ch}(E)$ the family of all nonempty and closed, bounded, relatively compact, chains, bounded chains and relatively compact chains of E respectively. Now we introduce the concept of partially measure of noncompactness in E on the lines of usual classical theory.

We accept the following definition of partially measure of noncompactness given in Dhage [6].

Definition 2.3. *A mapping $\mu^p : \mathcal{P}_{bd,ch}(E) \to \mathbb{R}_+ = [0, \infty)$ is said to be a partially measure of noncompactness in E if it satisfies the following conditions:*

1^o *$\emptyset \neq (\mu^p)^{-1}(\{0\}) \subset \mathcal{P}_{rcp,ch}(E)$,*

2^o *$\mu^p(\overline{C}) = \mu^p(C)$*

3^o *μ^p is nondecreasing, i.e., if $C_1 \subset C_2 \Rightarrow \mu^p(C_1) \leq \mu^p(C_2)$*

4^o *If $\{C_n\}$ is a sequence of closed chains from $\mathcal{P}_{bd,ch}(E)$ such that $C_{n+1} \subset C_n$ $(n = 1, 2, ...)$ and if $\lim\limits_{n\to\infty} \mu^p(C_n) = 0$, then the intersection set $\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty.*

The partially measure μ^p of noncompactness is called **sublinear** if it satisfies

5^o $\mu^p(C_1 + C_2) \leq \mu^p(C_1) + \mu^p(C_2)$ for all $C_1, C_2 \in \mathcal{P}_{bd,ch}(E)$ and

6^o $\mu^p(\lambda C) = |\lambda|\mu^p(C)$ for $\lambda \in \mathbb{R}$.

Remark 2.1. The family of sets described in 1^o is said to be *kernel of the measure of noncompactness μ^p* and is defined as

$$\ker\ \mu^p = \big\{C \in \mathcal{P}_{bd,ch}(E) \,\big|\, \mu^p(C) = 0\big\}.$$

Clearly, $\ker\ \mu^p \subset \mathcal{P}_{rcp,ch}(E)$. Observe that the intersection set C_∞ from condition 4^o is a member of the family $\ker\ \mu^p$. In fact, since $\mu^p(C_\infty) \leq \mu^p(C_n)$ for any n, we infer that $\mu^p(C_\infty) = 0$. This yields that $C_\infty \in \ker\ \mu^p$. This simple observation will be essential in our further investigations.

Definition 2.4. *A mapping $T : E \to E$ is called a partially k-set-contraction if there exists a constant $k > 0$ such that for any bounded chain C of E, $T(C)$ is a bounded chain and $\mu^p(T(C)) \leq k\,\mu^p(C)$.*

We need the following definition in what follows.

Definition 2.5 (Dhage [6]). *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* **compatible** *if $\{x_n\}$ is a monotone, that is, monotone a nondecreasing or monotone nonincreasing sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(X, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called* **Janhavi** *if the order relation and the metric or norm are compatible in it.*

The following applicable hybrid fixed point theorem for monotone mappings proved in Dhage [7] is the key tool for proving the main existence results of this paper.

Theorem 2.1 (Dhage [7]). *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in every compact chain C of E. Let $\mathcal{T} : S \to S$ be a partially continuous, nondecreasing and partially k-set-contraction with $k < 1$. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^*.*

Proof. The proof is given in Dhage [7] and so we omit the details. □

Remark 2.2. The regularity of E and the partial continuity of $\mathcal{T}$ in above Theorem 2.1 may be replaced with a stronger condition of the continuity of the operator $\mathcal{T}$ on E.

Remark 2.3. If the set S of solutions to the above operator equation is a chain, then all the solutions belonging to S are comparable. Further, if $\mu^p(S) > 0$, then $\mu^p(S) = \mu^p(\mathcal{T}S) \leq \psi(\mu^p(S)) < \mu^p(S)$ which is a contradiction. Consequently, $S \in \ker\ \mu^p$. This simple fact has been utilized in the study of qualitative properties of the dynamic systems under consideration. See Dhage [7, 8], Dhage and Dhage [9] and Dhage *et.al.* [10].

Remark 2.4. Suppose that the order relation $\preceq$ is introduced in E with the help of an order cone $\mathcal{K}$ which is a non-empty closed set $\mathcal{K}$ in E satisfying (i) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$, (ii) $\lambda \mathcal{K} \subseteq \mathcal{K}$ and (iii) $\{-\mathcal{K}\} \cap \mathcal{K} = \{0\}$ (cf. [12]). Then the order relation $\preceq$ in E is defined as $x \preceq y \iff y - x \in \mathcal{K}$. The element $x_0 \in E$ satisfying $x_0 \preceq \mathcal{T} x_0$ in above Theorem 2.1 is called a lower solution of the operator equation $x = \mathcal{T}x$. If the operator equation $x = \mathcal{T}x$ has more than one lower solution and the set of all these lower solutions are comparable, then the corresponding set S of the solutions to above operator equation is a chain and hence all solutions in S are comparable. To see this, let x_0 and y_0 be any two lower solutions of the above operator equation such that $x_0 \preceq y_0$ and let x^* and y^* respectively be the corresponding solutions under the conditions of Theorem 2.1. Now, by definition of $\preceq$, one has $y_0 - x_0 \in \mathcal{K}$ and from the monotone nondecreasing nature of $\mathcal{T}$ it follows that $\mathcal{T}^n y_0 - \mathcal{T}^n x_0 \in \mathcal{K}$. Since $\mathcal{K}$ is closed, we have that $y^* - x^* \in \mathcal{K}$ or $x^* \preceq y^*$.

For our purpose we introduce a handy tool for the partial measure of noncompactness in the space $BC(\mathbb{R}_+, \mathbb{R})$ which is useful in the study of the solutions of certain nonlinear integral equations. To define this partial measure of noncompactness, let us fix a nonempty and bounded chain X in the partially ordered Banach space $BC(\mathbb{R}_+, \mathbb{R})$ and a positive real number T. For a fixed element $x \in X$ and a real number $\epsilon \geq 0$ denote by $\omega^T(x, \epsilon)$ *the modulus of continuity* of the function x on the interval $[0, T]$ defined by

$$\omega^T(x, \epsilon) = \sup\{|x(t) - x(s)| : \ t, s \in [0, T], \ |t - s| \leq \epsilon\} .$$

Next, let us put

$$\omega^T(X, \epsilon) = \sup\{\omega^T(x, \epsilon) : \ x \in X\} ,$$

$$\omega_0^T(X) = \lim_{\epsilon \to 0} \omega^T(X, \epsilon)$$

and

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X) . \tag{2.1}$$

The partial Hausdorff measure of noncompactness β^p in the function space $C([0, T], \mathbb{R})$ of continuous real-valued functions defined on closed and bounded interval $[0, T]$, is very much useful in the applications to nonlinear differential and integral equations and it can be shown that

$$\beta^p(X) = \frac{1}{2}\omega_0^T(X)$$

for all bounded chain X in $C([0, T], \mathbb{R})$. Similarly, ω_0 is a handy tool of partial measure of noncompactness in the ordered Banach space $BC(\mathbb{R}_+, \mathbb{R})$ useful for practical applications to nonlinear differential and integral equations.

Now, for a fixed number $t \in \mathbb{R}_+$ and a fixed bounded chain X in $BC(\mathbb{R}_+, \mathbb{R})$, let us denote

$$X(t) = \{x(t) : x \in X\}.$$

Let

$$\delta_a(X(t)) = |X(t)| = \sup\{|x(t)| : \ x \in X\} ,$$

$$\delta_a^T(X(t)) = \sup_{t \geq T} \delta_a(X(t)) = \sup_{t \geq T} |X(t)|$$

and

$$\delta_a(X) = \lim_{T \to \infty} \delta_a^T(X(t)) = \limsup_{t \to \infty} |X(t)|. \tag{2.2}$$

The details of the function δ_a appear in Dhage [6]. Finally, let us consider the function μ_a^p defined on the family of bounded chains in $BC(\mathbb{R}_+, \mathbb{R})$ by the formula

$$\mu_a^p(X) = \omega_0(X) \, + \, \delta_a(X). \tag{2.3}$$

It can be shown that the function μ_a^p is a partial measure of noncompactness in the space $BC(\mathbb{R}_+, \mathbb{R})$. The components ω_0 and δ_a are called the characteristic values of the partial measure of noncompactness μ_a^p in $BC(\mathbb{R}_+, \mathbb{R})$.

Remark 2.5. The kernel ker μ_a^p of the partial measure of noncompactness μ_a^p consists of all nonempty and bounded chains X of the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X tends to zero at infinity. This particular characteristic of ker μ_a^p has been useful in establishing the local attractivity and local asymptotic stability of the comparable solutions for the fractional integral equations on $\mathbb{R}$.

3 ATTRACTIVITY AND STABILITY RESULTS

Our considerations will be placed in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of the all real functions $x = x(t)$ which are defined, continuous and bounded on $\mathbb{R}_+$. This space is equipped with the standard supremum norm

$$||x|| = \sup\{|x(t)| : \ t \in \mathbb{R}_+\}\ . \tag{3.1}$$

Define the order relation $\leq$ in $BC(\mathbb{R}_+, \mathbb{R})$ as follows. Let $x, y \in BC(\mathbb{R}_+, \mathbb{R})$. Then

$$x \leq y \iff x(t) \leq y(t) \tag{3.2}$$

for all $t \in \mathbb{R}_+$. It is clear that $(BC(\mathbb{R}_+, \mathbb{R}), \leq, \|\cdot\|)$ is a regular partially ordered Banach space which is also a lattice (cf. Nieto and Lopez [14]).

The following lemma follows immediately by an application of Arzellá-Ascoli theorem.

Lemma 3.1. *Let $\big(BC(\mathbb{R}_+, \mathbb{R}), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then the norm $\|\cdot\|$ and the order relation $\leq$ are compatible in every partially compact subset of $BC(\mathbb{R}_+, \mathbb{R})$.*

Proof. The proof of the lemma appears in Dhage [7]. Since it is not well-known, we give the details of it. Let S be a partially compact subset of $BC(\mathbb{R}_+, \mathbb{R})$ and let $\{x_n\}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots, \tag{$*$}$$

for each $t \in \mathbb{R}_+$.

Suppose that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}$ of the monotone real sequence $\{x_n(t)\}$ is convergent. By monotone characterization, the original sequence $\{x_n(t)\}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in \mathbb{R}_+$. This shows that the sequence $\{x_n\}$ converges point-wise to x in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\leq$ are compatible in S. This completes the proof. □

In order to introduce further concepts used in the paper let us assume that Ω is a nonempty chain of the space $BC(\mathbb{R}_+, \mathbb{R})$. Moreover, let Q be an operator defined on Ω with values in $BC(\mathbb{R}_+, \mathbb{R})$.

Consider the operator equation of the form

$$x(t) = Qx(t),\ t \in \mathbb{R}_+\ . \tag{3.3}$$

Definition 3.1. We say that comparable solutions of the equation (3.3) are **locally asymptotically stable** or **locally asymptotically attractive** to the line $x(t) = c$ for all $t \in \mathbb{R}_+$ if there exists an open ball $\mathcal{B}(x_0, r)$ in the space $BC(\mathbb{R}_+, \mathbb{R})$ such that for arbitrary comparable solution $x = x(t)$ of the equation (3.3) belonging to $\overline{\mathcal{B}}(x_0, r) \cap \Omega$ we have that

$$\lim_{t\to\infty} [x(t) - c] = 0\ . \tag{3.4}$$

In the case when limit (3.4) is uniform with respect to the set $\overline{\mathcal{B}}(x_0, r) \cap \Omega$, i.e. when for each $\epsilon > 0$ there exists $T > 0$ such that

$$|x(t) - c| \leq \epsilon \tag{3.5}$$

for all $x \in \overline{\mathcal{B}}(x_0, r) \cap \Omega$ being the comparable solutions of (3.3) and for $t \geq T$, we will say that the comparable solutions of the operator equation (3.3) are **uniformly locally asymptotically attractive** or **uniformly locally asymptotically stable** to the line $x(t) = c$ defined on $\mathbb{R}_+$.

The equation (1.1) will be considered under the following assumptions:

(H$_1$) The function $q\ :\ \mathbb{R}_+ \to \mathbb{R}$ is continuous and bounded. Moreover, $\lim\limits_{t\to\infty} q(t) = 0$.

(H$_2$) The function k is continuous and nonnegative on $\mathbb{R}_+ \times \mathbb{R}_+$.

(H$_3$) f and g define the functions $f, g\ :\ J \times \mathbb{R} \to \mathbb{R}_+$. Moreover, $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$.

(H$_4$) There exists a constant $L > 0$ such that

$$0 \leq f(t, x) - f(t, y) \leq L(x - y)$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$ with $x \geq y$.

(H$_5$) $g(t, x)$ is nondecreasing in x for each $t \in J$.

(H_6) There exists an element $u \in C(J,\mathbb{R})$ such that

$$u(t) \leq q(t) + \big[f(t,u(t))\big]\left(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ g(s,u(s)))\,ds\right)$$

for all $t \in J$.

(H_7) There exists a continuous function $b:\ \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(t,x) \leq b(t)$ for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Moreover, we assume that

$$\lim_{t\to\infty}\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ b(s)\,ds = 0\,.$$

Remark 3.1. *If we define the function $v:\mathbb{R}_+ \to \mathbb{R}_+$ by*

$$v(t) = \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ b(s)\,ds, \tag{3.6}$$

then it is continuous on $\mathbb{R}_+$ and which further in view of hypothesis (H_7) implies that the number $V = \sup_{t\geq 0} v(t)$ exists.

The hypotheses (H_1) through (H_7) are standard and have been widely used in the literature on nonlinear differential and integral equations. The hypothesis (H_3) is considered recently in Nieto and Lopez [14]. Now we formulate the main existence results for the integral equation (1.1) under above mentioned natural conditions.

Theorem 3.1. *Assume that the hypotheses (H_1) through (H_7) hold. Furthermore if $\frac{LV}{\Gamma(\alpha)} < 1$, then the fractional QFIE (1.1) has at least one solution x^* in the space $BC(\mathbb{R}_+,\mathbb{R})$ and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_{n+1}(t) = q(t) + \big[f(t,x_n(t))\big]\left(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ g(s,x_n(s))\,ds\right),\ t\in\mathbb{R}_+, \tag{3.7}$$

for each $n \in \mathbb{N}$ with $x_0 = u$ converges monotonically to x^*. Moreover, the comparable solutions of the QFIE (1.1) are uniformly locally asymptotically attractive and stable to zero on $\mathbb{R}_+$.

Proof. We seek the solutions of the HFIE (1.1) in the function space $BC(\mathbb{R}_+,\mathbb{R})$ of continuous and bounded real-valued functions defined on $\mathbb{R}_+$. Set $E = BC(\mathbb{R}_+,\mathbb{R})$. Then, in view of Lemma 3.1, every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define the operator Q defined on the space E by the formula

$$Qx(t) = q(t) + \big[f(t,x(t))\big]\left(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ g(s,x(s))\,ds\right) \tag{3.8}$$

for all $t \in \mathbb{R}_+$. Observe that in view of our assumptions, for any function $x \in E$, the function Qx is continuous on $\mathbb{R}_+$. As a result, Q defines a mapping $Q\ :\ E \to E$. We show that Q satisfies all the conditions of Theorem 2.1 on E. This will be achieved in a series of following steps:

Step I: *Q is a nondecreasing on E.*

Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (H_3)-(H_4), we obtain

$$\begin{aligned}
Qx(t) &= q(t) + \big[f(t,x(t))\big]\left(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ g(s,x(s))\,ds\right)\\
&\leq q(t) + \big[f(t,y(t))\big]\left(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ g(s,y(s))\,ds\right)\\
&= Qy(t)
\end{aligned}$$

for all $t \in \mathbb{R}_+$. This shows that Q is a nondecreasing operator on E.

Step II: *Q maps a closed and partially bounded set into itself.*

Define an open ball $\mathcal{B}(x_0,r)$, where $r = \dfrac{\|x_0\|+\|q\|}{1-\frac{LV}{\Gamma(\alpha)}}$. Let X be a chain in $\overline{\mathcal{B}}(x_0,r)$ and let $x \in X$ be arbitrary. If $x \geq \theta$, then for arbitrarily fixed $t \in \mathbb{R}_+$ we obtain:

$$\begin{aligned}
|x_0(t) - Qx(t)| &\leq |x_0(t)| + \|q\| + \big[|f(t,x(t))|\big]\times\\
&\qquad\times\left(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ |g(s,x(s))|\,ds\right)\\
&\leq |x_0(t)| + \|q\| + \big[|f(t,x(t)) - f(t,0)|\big]\times
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; b(s)\, ds \right) \\
\leq\;& |x_0(t)| + \|q\| + L\,|x(t)| \frac{v(t)}{\Gamma(\alpha)} \\
\leq\;& \|x_0\| + \|q\| + L\,\|x\| \frac{V}{\Gamma(\alpha)} \\
\leq\;& \|x_0\| + \|q\| + \frac{LVr}{\Gamma(\alpha)} \\
=\;& r. \qquad (3.9)
\end{aligned}
$$

Similarly, if $x \leq \theta$, then it can be shown that $|x_0(t) - Qx(t)| \leq r$ for all $t \in \mathbb{R}_+$. Taking the supremum over t, we obtain $\|x_0 - Qx\| \leq r$ for all $x \in X$. This means that the operator Q transforms any bounded chain X into a bounded chain QX in E. More precisely, we infer that the operator Q transforms the chain X belonging to $\overline{\mathcal{B}}(x_0, r)$ into the chain $Q(X)$ contained in the ball $\overline{\mathcal{B}}(x_0, r)$. As a result, Q defines a mapping $Q : \mathcal{P}_{ch}(\overline{\mathcal{B}}(x_0, r))) \to \mathcal{P}_{ch}(\overline{\mathcal{B}}(x_0, r)))$ and that Q is partially bounded on $S = \overline{\mathcal{B}}(x_0, r)$ into itself.

Step III: *Q is a partially continuous on S.*

Now we show that the operator Q is a partially continuous on the ball $\overline{\mathcal{B}}(x_0, r)$. To do this, let us fix an arbitrary $\epsilon > 0$ and take $x, y \in X \subset \overline{\mathcal{B}}(x_0, r)$ such that $x \geq y$ and $||x - y|| \leq \epsilon$. Then we get:

$$
\begin{aligned}
\big|Qx(t) - Qy(t)\big| \leq\;& \bigg| [f(t,x(t))] \Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; g(s,x(s))\, ds \Big) \\
&\quad - [f(t,y(t))] \Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; g(s,y(s))\, ds \Big) \bigg| \\
\leq\;& \bigg| [f(t,x(t))] \Big(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, g(s,x(s))\, ds \Big) \\
&\quad - [f(t,y(t))] \Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; g(s,y(s))\, ds \Big) \bigg| \\
&+ \bigg| [f(t,y(t))] \Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; g(s,x(s))\, ds \Big) \\
&\quad - [f(t,y(t))] \Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; g(s,y(s))\, ds \Big) \bigg| \\
\leq\;& \frac{1}{\Gamma(\alpha)} \big| f(t,x(t)) - f(t,y(t)) \big| \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \; b(s)\, ds \\
&+ \frac{2}{\Gamma(\alpha)} |f(t,y(t))| \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; b(s)\, ds \\
\leq\;& \frac{LV}{\Gamma(\alpha)} |x(t) - y(t)| + \frac{2Lr}{\Gamma(\alpha)} v(t).
\end{aligned}
$$

Hence, by virtue of hypothesis (B_6), we infer that there exists $T > 0$ such that $v(t) \leq \frac{\epsilon}{\frac{2Lr}{\Gamma(\alpha)}}$ for $t \geq T$. Thus, for $t \geq T$ we derive that

$$
|Qx(t) - Qy(t)| < \left(\frac{LV}{\Gamma(\alpha)} + 1 \right) \epsilon \,. \qquad (3.10)
$$

Further, let us assume that $t \in [0, T]$. Then, evaluating as above with the Similar arguments, we get:

$$
\begin{aligned}
\big|Qx(t) - Qy(t)\big| \leq\;& \bigg| [f(t,x(t))] \Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; g(s,x(s))\, ds \Big) \\
&\quad - [f(t,y(t))] \Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; g(s,y(s))\, ds \Big) \bigg| \\
\leq\;& \bigg| [f(t,x(t))] \Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; g(s,x(s))\, ds \Big) \\
&\quad - [f(t,y(t))] \Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; g(s,y(s))\, ds \Big) \bigg| \\
&+ \bigg| [f(t,y(t))] \Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} \;\; g(s,x(s))\, ds \Big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad -[f(t,y(t))]\Big(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ g(s,y(s))\,ds\Big)\Big| \\
&\le \frac{1}{\Gamma(\alpha)}\big|f(t,x(t))-f(t,y(t))\big|\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ b(s)\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}|f(t,y(t))|\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ |g(s,x(s))-g(s,y(s))|\,ds \\
&\le \frac{LV}{\Gamma(\alpha)}|x(t)-y(t)| + \frac{1}{\Gamma(\alpha)}|f(t,y(t))|\int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\ \omega_r^T(g,\epsilon)\,ds \\
&< \ \epsilon + \frac{C\,MT^p}{\Gamma(\alpha+1)}\,\omega_r^T(g,\epsilon)\ ,
\end{aligned}
\tag{3.11}
$$

where we have denoted

$$
\begin{aligned}
C &= \sup\{k(t,s)\ :\ t,s\in[0,T]\}, \\
M &= \sup\big\{f(t,y):\ t\in[0,T]\ \text{ and } y\in[-r,r]\big\}\ ,
\end{aligned}
$$

and

$$
\omega_r^T(g,\epsilon) = \sup\{|g(s,x)-g(s,y)|:\ t,s\in[0,T],\ x,y\in[-r,r],\ |x-y|\le\epsilon\}\ .
$$

Now, from the uniform continuity of the function $g(s,x)$ on the set $[0,T]\times[-r,r]$ we derive that $\omega_r^T(g,\epsilon)\to 0$ as $\epsilon\to 0$. Now, linking (3.10), (3.11) and the above established facts we conclude that the operator Q maps partially continuously the ball $\overline{\mathcal{B}}(x_0,r)$ into itself.

Step IV: *Q is a k-set-contraction w.r.t. the characteristic value ω_0.*

Further on let us take a chain X belonging to the ball $\mathcal{B}(x_0,r)$. Next, fix an arbitrary $T>0$ and $\epsilon>0$. Let us choose $x\in X$ and $t_1,t_2\in[0,T]$ with $|t_2-t_1|\le\epsilon$. Without loss of generality we may assume that $x(t_1)\ge x(t_2)$. Then, taking into account our assumptions, we get:

$$
\begin{aligned}
\big|Qx(t_1)-Qy(t_2)\big| &\le |q(t_1)-q(t_2)| \\
&\quad + \Big|[f(t_1,x(t_1))]\Big(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\frac{k(t_1,s)}{(t_1-s)^{1-\alpha}}\ g(s,x(s))\,ds\Big) \\
&\qquad -[f(t_2,y(t_2))]\Big(\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\frac{k(t_2,s)}{(t_2-s)^{1-\alpha}}\ g(s,y(s))\,ds\Big)\Big| \\
&\le |q(t_1)-q(t_2)| \\
&\quad + \Big|[f(t_1,x(t_1))]\Big(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\frac{k(t_1,s)}{(t_1-s)^{1-\alpha}}\ g(s,x(s))\,ds\Big) \\
&\qquad -[f(t_2,y(t_2))]\Big(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\frac{k(t_2,s)}{(t_2-s)^{1-\alpha}}\ g(s,y(s))\,ds\Big)\Big| \\
&\quad + \Big|[f(t_2,y(t_2))]\Big(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\frac{k(t_1,s)}{(t_1-s)^{1-\alpha}}\ g(s,x(s))\,ds\Big) \\
&\qquad -[f(t_2,y(t_2))]\Big(\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\frac{k(t_2,s)}{(t_2-s)^{1-\alpha}}\ g(s,y(s))\,ds\Big)\Big| \\
&\le |q(t_1)-q(t_2)| \\
&\quad + \frac{1}{\Gamma(\alpha)}\big|f(t_1,x(t_1))-f(t_2,x(t_2))\big|\Big(\int_0^{t_1}\frac{k(t_1,s)}{(t_1-s)^{1-\alpha}}\ b(s)\,ds\Big) \\
&\quad + \frac{1}{\Gamma(\alpha)}|f(t_2,y(t_2))|\Big|\int_0^{t_1}\frac{k(t_1,s)}{(t_1-s)^{1-\alpha}}\ |g(s,x(s))|\,ds \\
&\qquad\qquad -\int_0^{t_2}\frac{k(t_2,s)}{(t_2-s)^{1-\alpha}}\ |\,g(s,x(s))\,ds\Big| \\
&\quad + \Big|\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\frac{k(t_2,s)}{(t_2-s)^{1-\alpha}}\ g(s,x(s))\,ds\Big| \\
&\le |q(t_1)-q(t_2)| \\
&\quad + \big|f(t_1,x(t_1))-f(t_2,x(t_2))\big|v(t_1) \\
&\quad + \frac{M}{\Gamma(\alpha)}\Big|\int_0^{t_1}\frac{k(t_1,s)}{(t_1-s)^{1-\alpha}}\ g(s,x(s))\,ds - \int_0^{t_2}\frac{k(t_2,s)}{(t_2-s)^{1-\alpha}}\ g(s,x(s))\,ds\Big| \\
&\le |q(t_1)-q(t_2)| \\
&\quad + \frac{1}{\Gamma(\alpha)}\big|f(t_1,x(t_1))-f(t_2,x(t_2))\big|v(t_1)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)} \int_0^T \Big| \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} - \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \Big| b(s)\,ds \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t_1} \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \; g(s,x(s))\,ds \right| \\
\le{}& |q(t_1)-q(t_2)| \\
&+ \frac{V}{\Gamma(\alpha)} \big| f(t_1,x(t_1)) - f(t_2,x(t_2)) \big| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^T \Big| \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \; - \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \; \Big| b(s)\,ds \\
&+ \frac{G_r^T}{\Gamma(\alpha)} |t_1-t_2|,
\end{aligned}
\tag{3.12}
$$

where

$$G_r^T = \sup\{|\tilde{g}(t,s,x)| : t \in [0,T], s \in [0,T], x \in [-r,r]\}$$

which does exists in view of the fact that the function

$$\tilde{g}(t,s,x) = \frac{k(t,s)}{(t-s)^{1-\alpha}} \; g(s,x)$$

is continuous on compact $[0,T] \times [0,T] \times [-r,r]$. Now combining the inequalities (3.11) and (3.12) we obtain,

$$
\begin{aligned}
|Qx(t_2) - Qx(t_1)| &\le |q(t_1)-q(t_2)| \\
&\quad + \frac{V}{\Gamma(\alpha)} \big| f(t_1,x(t_1)) - f(t_2,x(t_1)) \big| \\
&\quad + \frac{LV}{\Gamma(\alpha)} \, |x(t_1) - x(t_2)| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^T \Big| \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \; - \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \; \Big| b(s)\,ds \\
&\quad + \frac{G_r^T}{\Gamma(\alpha)} |t_1-t_2| \\
&\le \omega^T(q,\epsilon) + \frac{LV}{\Gamma(\alpha)} \, \omega^T(x,\epsilon) + \frac{LV}{\Gamma(\alpha)} \omega_r^T(f,\epsilon) \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^T \Big| \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} \; - \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \; \Big| b(s)\,ds \\
&\quad + \frac{G_r^T}{\Gamma(\alpha)} |t_1-t_2|,
\end{aligned}
\tag{3.13}
$$

where we have denoted

$$\omega^T(q,\epsilon) = \sup\{|q(t_2) - q(t_1)| : \; t_1,t_2 \in [0,T], \; |t_2 - t_1| \le \epsilon\} \, ,$$

$$\omega^T(x,\epsilon) = \sup\{|x(t_2) - x(t_1)| : \; t_1,t_2 \in [0,T], \; |t_2 - t_1| \le \epsilon\}$$

and

$$\omega_r^T(f,\epsilon) = \sup\{|f(t_2,x) - f(t_1,x)| : \; t_1,t_2 \in [0,T], \; |t_2 - t_1| \le \epsilon, \; x \in [-r,r]\} \, .$$

From the above estimate we derive the following one:

$$
\begin{aligned}
\omega^T(Q(X),\epsilon) &\le \omega^T(q,\epsilon) + \frac{LV}{\Gamma(\alpha)} \, \omega^T(X,\epsilon) + \frac{LV}{\Gamma(\alpha)} \omega_r^T(f,\epsilon) \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^T \Big| \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} - \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \Big| b(s)\,ds \\
&\quad + \frac{G_r^T}{\Gamma(\alpha)} \epsilon.
\end{aligned}
\tag{3.14}
$$

Observe that $\omega_r^T(f,\epsilon) \to 0$ and $\Big| \frac{k(t_1,s)}{(t_1-s)^{1-\alpha}} - \frac{k(t_2,s)}{(t_2-s)^{1-\alpha}} \Big| \to 0$ as $\epsilon \to 0$, which is a simple consequence of the uniform continuity of the functions f and $\dfrac{k(t,s)}{(t-s)^{1-\alpha}}$ on the sets $[0,T] \times [-r,r]$ and $[0,T] \times [0,\beta_T]$ respectively. Moreover, from the uniform continuity of q on $[0,T]$, it follows that $\omega^T(q,\epsilon) \to 0$ as $\epsilon \to 0$. Thus, linking the established facts with the estimate (3.14) we get

$$\omega_0^T(Q(X)) \le \frac{LV}{\Gamma(\alpha)} \omega_0^T(X) \, .$$

Consequently, we obtain

$$\omega_0(Q(X)) \leq \frac{LV}{\Gamma(\alpha)}\, \omega_0(X)\,. \tag{3.15}$$

Step V: *Q is a k-set-contraction w.r.t. characteristic value δ_c.*

Next, we show that Q is k-set-contraction with respect to the characteristic value δ_a. Now, taking into account our assumptions, for arbitrarily fixed $t \in \mathbb{R}_+$ and for $x \in X$ with $x \geq 0$, we deduce the following estimate:

$$\begin{aligned} |(Qx)(t)| &\leq |q(t)| + |f(t,x(t)) - f(t,0)| \left(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} b(s)\, ds \right) \\ &\leq |q(t)| + L\,|x(t)| \frac{v(t)}{\Gamma(\alpha)} \\ &\leq |q(t)| + \frac{LV}{\Gamma(\alpha)} |x(t)|. \end{aligned}$$

From the above inequality it follows that

$$|QX(t)| \leq |q(t)| + \frac{LV}{\Gamma(\alpha)} |X(t)|$$

for each $t \in \mathbb{R}_+$. Therefore, taking limit superior over $t \to \infty$, we obtain

$$\begin{aligned} \delta_a(QX) &= \limsup_{t\to\infty} |Q(X(t))| \\ &\leq \frac{LV}{\Gamma(\alpha)} \limsup_{t\to\infty} |X(t)| \\ &= \frac{LV}{\Gamma(\alpha)}\, \delta_a(X). \end{aligned} \tag{3.16}$$

Step VI: *Q is a partially k-set-contraction on S.*

Further, using the measure of noncompactness μ_a^p defined by the formula (2.3) and keeping in mind the estimates (3.15) and (3.16), we obtain

$$\begin{aligned} \mu_c^p(QX) &= \omega_0(QX) \,+\, \delta_a(QX) \\ &\leq \frac{LV}{\Gamma(\alpha)}\, \omega_0(X) \,+\, \frac{LV}{\Gamma(\alpha)}\, \delta_a(X) \\ &= \frac{LV}{\Gamma(\alpha)}\, \mu_a^p(X). \end{aligned}$$

This shows that Q is a partially nonlinear k-set-contraction on S with $k = \frac{LV}{\Gamma(\alpha)} < 1$. Again, by hypothesis ($H_5$), there exists an element $x_0 = u \in S$ such that $x_0 \leq Qx_0$, that is, x_0 is a lower solution of the QFIE (1.1) defined on $\mathbb{R}_+$.

Thus Q satisfies all the conditions of Theorem 2.1 on S. Hence we apply it to the operator equation $Qx = x$ and deduce that the operator Q has a fixed point x^* in the ball $\overline{\mathcal{B}}(x_0, r)$. This further implies that x^* is a solution of the fractional integral equation (1.1) and the sequence $\{x_n\}$ of successive approximations defined by

$$x_{n+1}(t) = q(t) + \left[f(t, x_n(t))\right] \left(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\; g(s, x_n(s))\, ds \right)$$

for each $t \in \mathbb{R}_+$ converges monotonically to x^*. Moreover, taking into account that the image of every chain X under the operator Q is again a chain $Q(X)$ contained in the ball $\overline{\mathcal{B}}(x_0, r)$ we infer that the set $\mathcal{F}(Q)$ of all fixed points of Q is contained in $\overline{\mathcal{B}}(x_0, r)$. If the set $\mathcal{F}(Q)$ contains all comparable solutions of the equation (1.1), then we conclude from Remark 2.3 that the set $\mathcal{F}(Q)$ belongs to the family ker μ_a^p. Now, taking into account the description of sets belonging to ker μ_a^p (given in Section 2) we deduce that all comparable solutions of the equation (1.1) are uniformly locally ultimately attractive on $\mathbb{R}_+$. This completes the proof. $\square$

Remark 3.2. The conclusion of Theorem 3.1 also remains true if we replace the hypothesis (H_5) with the following one:

(H_5') There exists an element $u \in C(\mathbb{R}_+, \mathbb{R})$ such that

$$u(t) \geq q(t) + \left[f(t, u(t))\right] \left(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}}\; g(s, u(s))\, ds \right)$$

for all $t \in \mathbb{R}_+$.

The proof under this new hypothesis is similar to Theorem 3.1 and the conclusion again follows by an application of Theorem 2.1.

Below we indicate an example for the realization of the abstract theory we have developed in the previous part of the paper.

Example 3.1. Consider the nonlinear quadratic fractional integral equation,

$$x(t) = \frac{1+t}{1+t^2} + \big[f(t,x(t))\big]\left(\frac{1}{\Gamma(3/2)}\int_0^t \frac{(t-s)^{1/2}}{t^2+1}\; g(s,x(s))\,ds\right) \tag{3.17}$$

for all $t \in \mathbb{R}_+$, where $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are functions defined by

$$f(t,x) = \begin{cases} 0, & \text{if } \; x \le 0, \\ \dfrac{\sin x}{20}, & \text{if } \; 0 < x \le \dfrac{\pi}{2}, \\ \dfrac{(\pi+2)x}{20\pi(1+x)}, & \text{if } \; x > \dfrac{\pi}{2}, \end{cases}$$

and

$$g(t,x) = \begin{cases} 1, & \text{if } \; x \le 0, \\ x^2+1, & \text{if } \; 0 < x \le 1, \\ \dfrac{4x}{1+x}, & \text{if } \; x > 1. \end{cases}$$

We shall show that all the of Theorem 3.1 are satisfies by the functions involved in QFIE (3.17). Here, $q(t) = \dfrac{1+t}{1+t^2}$ so that q is continuous and bounded on $\mathbb{R}_+$ with bound equal to unity. Again, $\lim\limits_{t\to\infty} q(t) = 0$. Thus, hypothesis ($\mathrm{H}_1$) of Theorem 3.1 is satisfied. Again, here the kernel function $k(t,s)$ is given by $k(t,s) = \frac{1}{t^2+1}$. Obviously k is continuous and nonnegative on $\mathbb{R}_+ \times \mathbb{R}_+$ and so (H_2) holds. Next, f and g define the continuous functions $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ and $f(t,x)$ and $g(t,x)$ are nondecreasing in x for each $t \in \mathbb{R}_+$. Moreover, $f(t,0) = 0$. So the hypotheses (H_3) and (H_5) are satisfied.

Now, we show that f is partially Lipschitz on $\mathbb{R}_+ \times \mathbb{R}$. Let $x, y \in (-\infty, 0]$ with $x \ge y$. Then,

$$0 \le f(t,x) - f(t,y) \le L(x-y)$$

for all real numbers $L > 0$. If $x, y \in [0, \frac{\pi}{2}]$ and $x \ge y$, then

$$0 \le f(t,x) - f(t,y) \le \frac{1}{20}(x-y).$$

Again, if $x, y \in \mathbb{R}_+$ with $x \ge y$, then

$$0 \le f(t,x) - f(t,y) \le \frac{\pi+2}{20\pi}(x-y).$$

Since $\dfrac{\pi+2}{20\pi} > \dfrac{1}{20}$, one has

$$0 \le f(t,x) - f(t,y) \le \frac{\pi+2}{20\pi}(x-y).$$

for all $x, y \in \mathbb{R}$ and so, hypothesis (H_4) is held.

Furthermore, $\alpha = \frac{3}{2}$ and $g(t,x)| \le 4$ for all $t \in \mathbb{R}_+$ and $\mathbb{R}$. Therefore,

$$v(t) = \int_0^t \frac{(t-s)^{\frac{1}{2}}}{t^2+1}\cdot 4\,ds = \frac{8}{3}\cdot\frac{t^{\frac{3}{2}}}{t^2+1}.$$

Therefore,

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{8}{3}\,\frac{t^{\frac{3}{2}}}{t^2+1} = 0.$$

As a result the hypothesis (H_7) is held. Furthermore,

$$\frac{LV}{\Gamma(\alpha)} = \left(\frac{\pi+2}{20\pi}\right)\frac{4}{3\Gamma(3/2)} < 1.$$

Finally, it is easy to prove thawt $u \equiv 0$ is a lower solution of the QFIE (3.16) on $\mathbb{R}_+$ and hence the hypothesis (H_6) is held. Thus all the conditions of Theorem 3.1 are satisfied and by a direct application, we conclude that the QFIE (3.17) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined by

$$x_{n+1}(t) = \frac{1+t}{1+t^2} + \big[f(t,x_n(t))\big]\left(\frac{1}{\Gamma(3/2)}\int_0^t \frac{(t-s)^{1/2}}{t^2+1}\; g(s,x_n(s))\,ds\right),\; t \in J, \tag{3.18}$$

converges monotonically to x^*, where $x_0 = 0$. Moreover, the comparable solutions of the QFIE (3.17) are uniformly asymptotically attractive and stable to 0 defined on $\mathbb{R}_+$.

4 CONCLUSION

In this paper we have been able to weaken the Lipschitz condition to partially Lipschitz condition which otherwise is considered to be a very strong condition in the existence theory of nonlinear differential and integral equations. However, in such situations we need an additional assumption of the monotonicity on the nonlinearities involved in the considered fractional integral equation in order to guarantee the required characterization of the asymptotic attractivity or asymptotic stability of the comparable solutions. The advantage of the present approach over the previous ones lies in the fact that we have been able to develop an algorithm for the solutions of the considered integral equations which otherwise is not possible via classical approach of measure of noncompactness discussed in Banas and Goebel [2]. Finally, while concluding this paper we mention that the results presented here are of local nature, however analogous study can also be made for global asymptotic attractivity and stability using the similar arguments with appropriate modifications and some of the results in this direction will be elsewhere.

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