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Existence results for Graph Differential Equations through its Associated Matrix Differential Equations

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ABSTRACT: A network can be represented by graph which is isomorphic to its adjacency matrix. Thus the analysis of networks involving rate of change with respect to time reduces to the study of graph differential equations and its associated matrix differential equations. In this paper we develop the Peano's and Picard's existence theorem for graph differential equations through its associated matrix differential equations.

1 INTRODUCTION

In any physical phenomena interconnections between the components of the considered system arise naturally. These interconnections can be well represented by a graph. In social structures these inter connections vary with time and hence there interconnections are better represented in a graph that changes with time. Hence in [D.D.Siljak, 2008] graph functions have been introduced.A natural extension would be to consider the rate of change of these graphs with respect to time. This has been introduced by Siljak in [V.Lakshmikantham et al., 1969]. In [J.Vasundhara Devi et al., 2013] an attempt had been made to systematically introduce and study the concepts of graphs that lead to a graph differential equation (GDE) and an existence result was developed through the monotone iterative technique.

In [J.Vasundhara Devi et al., 2013] it has been observed that a simple digraph is not useful in applications and a concept of pseudo simple graph had been introduced along with other concepts like the product of graphs.

In this paper we first consider a matrix differential equation (MDE) and study an existence result of a Peano type and consider the extension of solution to the boundary of the given domain. Next we use this result to obtain the corresponding results to the given graph differential equation.

2 PRELIMINARIES

In this section, we give certain definitions,notations,results and preliminary facts related to GDEs that are required to study the main results in the problem.

2.1 Definition : Pseudo simple graph

A simple graph having loops is called as a pseudo simple graph.

Analogous to theory of directed simple graphs developed in [J.Vasundhara Devi et al., 2013] we proceed to develop the results in this set up. We avoid the details for fear of repetition.

Let $v_1, v_2, ...v_N$ be N vertices, where N is any positive integer. Let D_N be the set of all weighted directed pseudo simple graphs D=(V, E). Then $(D_N, +, .)$ is a linear space w.r.t the operations + and . defined in [J.Vasundhara Devi et al., 2013].

Let the set of all corresponding adjacency matrices be E_N. Then $(E_N, +, .)$ is a matrix linear space where '+' denotes matrix addition and '.' denotes scalar multiplication. With this basic structure defined, the comparison theorems, existence and uniqueness results of a solution of a MDE and the corresponding GDE follow as in [J.Vasundhara Devi et al., 2013].

2.2 Definition: Continuous and differentiable matrix

(1) A matrix $E(t) = (e_{ij})_{N\times N}$ is said to be continuous if and only if each entry $e_{ij}(t)$ is continuous for all $i, j = 1, 2, \ldots, N$.
(2) A continuous matrix $E(t)$ is said to be differentiable if and only if each entry $e_{ij}(t)$ is differentiable for all $i, j = 1, 2, \ldots, N$. The derivative of E(t) (if exists) is denoted by E' and is given by $E'(t) = (e'_{ij})_{N\times N}$.

2.3 Definition : Continuous and differentiable graph

Let $D : I \to D_N$ be a graph and $E : I \to R^{n\times n}$ be its associated adjacency matrix then
(1) D(t) is said to be continuous if and only if E(t) is continuous.
(2) D(t) is said to be differentiable if and only if E(t) is differentiable.

If for any graph D the corresponding adjacency matrix is differentiable then we say that D is differentiable and the derivative of D(if exists) is denote by D'.

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Consider the initial value problem

$$D' = \mathcal{G}(t, D), \quad D(t_0) = D_0 \tag{1}$$

Let E, E_0 be adjacency matrices corresponding to any graph D and the initial graph D_0.

Then the MDE is given by

$$E' = F(t, E), \quad E(t_0) = E_0 \tag{2}$$

where F(t,E) is the adjacency matrix function corresponding to $\mathcal{G}(t, D)$.

2.4 Definition : Solution of a Matrix Differential Equation

Any continuous differentiable matrix function E(t) is said to be a solution of (2), if and only if it satisfies (2).

2.5 Definition: Solution of a Graph Differential Equation

By a solution of GDE (1) we mean the graph function D(t) corresponding to the matrix solution E(t) of the MDE (2).

In order to obtain a unique solution of (1) we use the corresponding adjacency MDE. As there exists an isomorphism between graphs and matrices, the solution obtained for the MDE will be a solution of the corresponding GDE.

2.6 Definition: Convergence of a sequence of Matrices

Let $\{E_n\}$ be a sequence of matrices then $\mathrm{E_n}$ converges to E if and only if given $\epsilon > 0$ there exist $\mathrm{n} \geq \mathrm{N}$ such that $||\mathrm{E_n} - \mathrm{E}|| \leq \epsilon \; for \; all \quad \mathrm{n} \geq \mathrm{N}$, for all i, j . This means $\mathrm{e_{n_{ij}}} \to e_{ij}$ for all $1 \leq i, j \leq N$

2.7 Definition:

Consider two matrices A and B of order N. We say that $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i, j = 1, 2, \ldots, N$.

With the necessary preliminaries in place, we proceed to the next section to develop the main results.

3 Existence Result

In this section we plan to first prove an existence result for the MDE obtained from the given GDE and then using the solution of the MDE we construct the graph solution for the given GDE.

In order to prove the existence result for MDE we need Ascoli's theorem extended to matrices, that is for $f \in \mathbb{R}^{n \times n}$. As the proof is parallel to the standard result we only state the Ascoli's Theorem for matrices below.

3.1 Theorem

Let $F = \{f | f : [t_0, T] \to \mathbb{R}^{n \times n}\}$ be a family of functions which is equicontinuous and uniformly bounded. Then there exists a subsequence $\{f_k\}$, $k = 1, 2, ...$ which is uniformly convergent on the interval $[t_0, T]$.

Consider the nonlinear GDE given by (1) where $\mathcal{G}(t, D) \in D_N$ and g_{ij} is the weight of the edge from v_j to v_i of the graph $\mathcal{G}(t, D)$.

Let $F(t, E)$ be the adjacency matrix function corresponding to the graph $\mathcal{G}(t, D)$. Clearly F is nonlinear. Now consider the MDE be given by

$$\left.\begin{array}{rcl} E' & = & F(t, E) \\ E(t_0) & = & E_0 \end{array}\right\} \tag{3}$$

Where E_0 is the initial adjacency matrix corresponding to initial graph D_0, $F(t, E) \in \mathbb{R}^{n \times n}$ and $E \in \mathbb{R}^{n \times n}$ Let us introduce the following notation.

R_0 ={(t, E): $|t - t_0| \leq a \; and \; ||E - E_0|| \leq b\}$ be a closed compact set in $[t_0, T] \times \mathbb{R}^{n \times n}$ where $||E|| = Max|e_{ij}|$ and $E = (e_{ij})_{n \times n}$.

We shall now state and prove the following theorem parallel to the Peano's theorem for ordinary differential equation in $\mathbb{R}^n$.

3.2 Peano's Theorem

Let $F \in C[[t_0, T], \mathbb{R}^{n \times n}]$ and $||F(t, E)|| \leq M$ on R_0. Then the initial value problem of the MDE possesses at least one solution $E(t)$ on $t_0 \leq t \leq t_0 + \alpha$, where $\alpha = min(a, \frac{b}{M})$

Proof.

We consider an interval $[t_0 - \delta, t_0]$ for small $\delta > 0$ and choose a continuous differentiable function $E_0(t)$ defined on $[t_0 - \delta, t_0]$ such that

$E_0(t_0) = E_0, ||E_0(t) - E_0|| \leq b$ and $||E_0'(t)|| \leq M$

Next, take ϵ_1 such that $0 < \epsilon_1 < \delta$ and define the function $E_{\epsilon_1}(t)$ as follows

$$E_{\epsilon_1}(t) = \begin{cases} E_0, & t \in [t_0 - \delta, t_0] \\ E_0 + \int_{t_0}^{t} F(s, E_{\epsilon_1}(s - \epsilon_1))ds, & t \in [t_0, t_0 + \alpha_1] \end{cases} \tag{4}$$

where $\alpha_1 = min(\alpha, \epsilon_1)$. Clearly $E_{\epsilon_1}(t)$ is a differentiable function,

and

$$||E_{\epsilon_1}(t) - E_0|| \le b, \quad ||E'_{\epsilon_1}(t)|| \le M, \quad [t_0 - \delta, t_0 + \alpha_1] \ \} \tag{5}$$

Now if $\alpha_1 = \alpha$, we are through. If $\alpha_1 < \alpha$ then using (4) we extend the function to the interval $[t_0 - \delta, t_0 + \alpha_2]$ as follows

$$E_{\epsilon_1}(t) = \begin{cases} E_0, & t \in [t_0 - \delta, t_0] \\ E_0 + \int_{t_0}^{t} F(s, E_{\epsilon_1}(s - \epsilon_1))ds, & t \in [t_0, t_0 + \alpha_2] \end{cases} \tag{6}$$

where $\alpha_2 = min(\alpha, 2\epsilon_1)$ such that (5) holds on the interval $[t_0 - \delta, t_0 + \alpha_2]$. Proceeding in this fashion, we can define $E_{\epsilon_1}(t)$ over the interval $[t_0 - \delta, t_0 + \alpha]$ such that $E_{\epsilon_1}(t)$ is continuously differentiable and satisfies the relation (5) on the interval $[t_0 - \delta, t_0 + \alpha]$. It can be easily verified that $||E'_{\epsilon_1}(t)|| \le M$ on $[t_0 - \delta, t_0 + \alpha]$. Using ϵ we obtain a family of functions $E_\epsilon(t)$ of matrix functions,where $E_\epsilon(t)$ defined as in (4) and satisfies (5).

Now the family of functions $\{E_\epsilon(t)\}$ is equicontinuous and uniformly bounded. Thus applying Ascoli's Theorem for a family of matrix functions. We can conclude that there exists a sequence of functions $\{E_{\epsilon_n}(t)\}, n = 1, 2, 3, ...$ which is uniformly convergent. Note $\{\epsilon_n\} \to 0$ as $n \to \infty$

Let the sequence $E_{\epsilon_n}(t)$ be converge uniformly to $E(t)$ on $[t_0 - \delta, t_0 + \alpha]$.

Since F is a uniformly continuous function on R, we obtain that

$F(t, E_{\epsilon_n}(t - \epsilon_n))$ converges uniformly to $F(t, E(t))$ as $n \to \infty$. Hence term-by-term integration of (4) with $\epsilon = \epsilon_n$, $\alpha_1 = \alpha$ yields

$E(t) = E_0 + \int_{t_0}^{t} F(s, E(s))ds$

This proves that $E(t)$ is a solution of (3) and the proof is complete.

3.3 Corollary

Let R be an open $(t, E) \subset [[t_0, t_0 + a] \times \mathbb{R}^{n \times n}]$ and R_0 be a compact subset of R. Suppose that $F \in C[\mathbb{R}, \mathbb{R}^{n \times n}]$ and $||F(t, E)|| < M$ in R then there exists and $\alpha = \alpha(E, E_0, M)$ such that $(t_0, F_0) \in R_0$ the MDE (2) has a solution, and every solution exists on $[t_0, t_0 + \alpha]$.

3.4 Theorem

Let $\mathcal{G}$ in the IVP for GDE (1) be continuous and bounded. Then the IVP for GDE (1) possesses at least one solution.

Proof.

Let D be any graph and E be its corresponding adjacency matrix. Construct the adjacency matrix E_0 corresponding the given initial graph D_0. Let $F(t, E)$ be the adjacency matrix corresponding $\mathcal{G}$ in (1). Since $\mathcal{G}$ is continuous it implies that F is also continuous and F is also bounded. Thus from the Theorem 3.2, there exists least one solution $E(t)$ for the MDE.

$E' = F(t, E), E(t_0) = E_0$. Now the corresponding graph function $D(t)$ is a solution for the GDE (1).

We now state and prove following lemma which is necessary to prove a theorem dealing with extending the solutions of a MDE to the domain.

3.5 Lemma

Let $F \in C[R, \mathbb{R}^{n \times n}]$ where R is an open (t, E) set in $R \times \mathbb{R}^{n \times n}$. Let $E(t)$ be solution of (2) on any interval $t_0 \le t < a$, $a < \infty$. Assume that there exist a sequence $\{t_k\}$ such that $t_0 \le t_k \to \infty$ as $k \to \infty$

$$lim_{k \to \infty} E(t_k) = E^0 \tag{7}$$

(7) exists if $F(t, E)$ is a bounded on the intersection of R and neighborhood of (a, E^0) then $\lim_{k \to \infty} E(t_k) = E^0$. If, in addition $F(t, E)$ is continuous at (a, E^0), then $E(t)$ is continuously differentiable on $[t_0, a]$ and is a solution of (2) on $[t_0, a]$.

Proof.

As we are interested in the behavior of the solution $E(t)$ of (2) at the boundary point $t = a$. Let us consider the region R as follows choose $\epsilon > 0$ sufficiently small and set

$\bar{R} = \{(t, E) : 0 < a - t < \epsilon, ||E - E^0|| \le \epsilon\}$

Since F is bounded on $R \cap \bar{R}$ there exist $M(\epsilon)$ sufficiently large such that $||F(t, E)|| < M(\epsilon)$ for $(t, E) \in R \cap \bar{R}$ next using the hypothesis, we get that for sufficiently large k, given $\epsilon > 0$ there exists $\eta > 0$ such that $||E(t_k) - E^0|| < \frac{\epsilon}{2}$ and $0 < a - t_k < \eta$ choose $\eta = \frac{\epsilon}{2M(\epsilon)}$

For $t_k \leq t < a$, consider $||E(t) - E^0|| \leq ||E(t) - E(t_k)|| + ||E(t_k) - E^0||$ for a sufficiently large k, we get by considering

$$\begin{aligned} ||E(t) - E(t_k)|| &\leq ||\int_{t_k}^{t} F(s, E(s))ds|| \\ &\leq M(\epsilon)(t - t_k) \\ &< M(\epsilon)(a - t_k) \\ &< \frac{\epsilon}{2} \end{aligned} \quad (8)$$

Now suppose (8) does not hold then there exists a t_1 such that $t_k < t_1 < a$ such that $||E(t_1) - E(t_k)|| = M(\epsilon)(a - t_k) \leq \frac{\epsilon}{2}$

From this relation we can still conclude that for $t_k \leq t < t_1$

$||E(t) - E^0|| \leq \frac{\epsilon}{2} + ||E(t_k) - E^0|| < \epsilon$

which further yields that for $t_k \leq t < t_1$, $||F'(t)|| \leq M(\epsilon)$

Hence $||E(t_1) - E(t_k)|| \leq M(\epsilon)(t_1 - t_k) < M(\epsilon)(a - t_k)$. Thus (8) holds, which implies that (7) holds now since F is continuous and $E(t) \to E^0$ as $t \to a$ Thus $E(t)$ is a solution of (2) as $[t_0, a]$ and hence thus the proof is complete.

3.6 Theorem

Let R be an open set in $R \times \mathbb{R}^{n\times n}$ and let

$F \in C[R, \mathbb{R}^{n\times n}]$ and $E(t)$ be a solution of (2) on some interval $t_0 \leq t \leq a_0$. Then $E(t)$ can be extended as a solution to the boundary of R.

Proof.

Let $R_1, R_2, ..., R_n$ be open subsets of R such that $\bar{R_1}, \bar{R_2}, \bar{R_3}, ..., \bar{R_n}$ are compact and $\bar{R_n} \subset R_{n+1}$

Suppose $(t_0, E_0) \in \bar{R_n}$ for some n, since F is continuous and from Corollary 3.3, we have that there exists an $\epsilon_n > 0$ such that all solutions of the IVP (2) exist on $[t_0, t_0 + \epsilon_n]$. Now consider the solution $E(t)$ of IVP(2) existing on $t_0 \leq t \leq a_0$ and take $(a_0, E(a_0))$. Choose n_1 so large that $(a_0, E(a_0)) \in \bar{R_n}$. Then by Corollary 3.3, $E(t)$ can be extended over the interval $(a_0, a_0 + \epsilon_n)$. if $(a_0 + \epsilon_{n_1}, E(a_0 + \epsilon_{n_1})) \in \bar{R_{n_1}}$, then $E(t)$ can be further extended over the interval $[a_0 + \epsilon_{n_1}, a_0 + 2\epsilon_{n_1}]$ repeating this argument finite number of times. We can extend the solution $E(t)$ over the interval $t_0 \leq t \leq a_1$, where $a_1 = a_0 + N_1\epsilon_{n_1}$ where N_1 an integer ≥ 1 such that $(a_1, E(a_1)) \notin \bar{R_{n_2}}$

Next, we choose n_2 so large that $(a_1, E(a_1)) \in \bar{R_{n_2}}$ proceeding as before, we can find an integer $N_2 \geq 1$ such that $E(t)$ can be extended on the interval $[t_0, a_2]$ where $a_2 = a_1 + N_2\epsilon_{n_2}$ and$(a_2, E(a_2)) \notin \bar{R_{n_2}}$

Working as earlier , we obtain a sequence of integers $n_1 < n_2 < ...$ and numbers $a_0 < a_1 < a_2 < ...$ such that $E(t)$ can be extended over the interval $[t_0, a]$ where $\lim_{k\to\infty} a_k = a$ and that $(a_k, E(a_k)) \notin \bar{R_{n_k}}$. We now consider the sequence of points $(a_k, E(a_k))$ this sequence is either unbounded or has a limit point on the boundary of R. From Lemma 3.5, we have that no limit point of $(t_k, E(t_k))$ is an interior point of R as $t_k \to a$, hence $E(t)$ tends to the boundary of R as $t \to a$.

Thus the proof is complete.

Remark : Using the above Theorem 3.6, we can easily show that any solution $\mathcal{G}(t)$ of the GDE (1) exists on $t_0 \leq t \leq a$, $a < \infty$ can be extended to the boundary of the considered domain provided $\mathcal{G}$ is continuous.

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