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Farkas' Lemma, Gale's Theorem, and Linear Programming: the Infinite Case in an Algebraic Way

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ABSTRACT: We study a problem of linear programming in the setting of a vector space over a linearly ordered (possibly skew) field. The dimension of the space may be infinite. The objective function is a linear mapping into another linearly ordered vector space over the same field. In that algebraic setting, we recall known results: Farkas' Lemma, Gale's Theorem of the alternative, and the Duality Theorem for linear programming with finite number of linear constraints. Given that "semi-infinite" case, i.e. results for finite systems of linear inequalities in an infinite-dimensional space, we are motivated to consider the infinite case: infinite systems of linear inequalities in an infinite-dimensional space. Given such a system, we assume that only a finite number of the left-hand sides is non-zero at a point. We shall also assume a certain constraint qualification (CQ), presenting counterexamples violating the (CQ). Then, in the described setting, we formulate an infinite variant of Farkas' Lemma along with an infinite variant of Gale's Theorem of the alternative. Finally, we formulate the problem of an infinite linear programming, its dual problem, and the Duality Theorem for the problems.

1 INTRODUCTION

There are many generalizations of Farkas' Lemma and Duality Theorem for linear programming in the literature (Anderson & Nash, 1987; Goberna & López, 1998). In the following section, we recall a particular generalization due to Bartl (2007). We shall introduce some notation and concepts first.

Let F be a linearly ordered (possibly skew) field. (A field is skew if and only if it is not commutative. In other words, the field F may or may not be commutative.) The ordering of the field F is a binary relation "$\leq$" such that, for all $\lambda, \mu \in F$,

$$\lambda \leq \mu \iff \lambda - \mu \leq 0$$

and

$$\begin{gathered} \lambda \leq 0 \;\vee\; \lambda \geq 0\,, \\ \lambda \leq 0 \;\wedge\; \lambda \geq 0 \implies \lambda = 0\,, \\ \lambda \geq 0 \;\wedge\; \mu \geq 0 \implies \lambda + \mu \geq 0\,, \\ \lambda \geq 0 \;\wedge\; \mu \geq 0 \implies \lambda\mu \geq 0\,, \end{gathered}$$

where we have used the usual convention that $\lambda \geq \mu$ if and only if $\mu \leq \lambda$. The field of the real numbers $\mathbb{R}$ or that of the rational numbers $\mathbb{Q}$ with the usual ordering is an example of a linearly ordered commutative field. Linearly ordered skew fields also exists; an example of such a field was given already by Hilbert in 1901, see Cohn (1995, Notes and comments to Chapter 1, p. 45, with Sections 2.1 and 2.3, pp. 47–50 and 66) and Lam (1991, Example 1.7, p. 10, and above Proposition 18.7, p. 288).

Let W be a vector space over the field F. No additional structure (such as topology) is assumed on the space W, whose dimension may be finite or infinite. For example, if $F = \mathbb{R}$, then W can be $\mathbb{R}^n$, finite-dimensional, or $\mathcal{C}_{[0,1]}$, the space of real continuous functions on the closed interval $[0,1]$, or another functional space. Considering a problem of linear optimization (or programming), the space W will be the "base" space in which we shall work:

Let $A\colon W \to F^m$ be a linear mapping and let $\boldsymbol{b} \in F^m$ be a column vector. Then

$$Ax \leq \boldsymbol{b}$$

is a finite system of linear inequalities, which circumscribes the set of the feasible solutions. For example, if $F = \mathbb{R}$ and $W = \mathbb{R}^n$, then A is induced by a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$.

Let V be a linearly ordered vector space over the linearly ordered (possibly skew) field F. The ordering of the space is a binary relation "$\preceq$" such that, for all $u, v \in V$,

$$u \preceq v \iff u - v \preceq 0$$

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and, for all $\lambda \in F$ and $u, v \in V$, it holds

$$\begin{aligned} u \preceq 0 \ \vee \ u \succeq 0\,, \\ u \preceq 0 \ \wedge \ u \succeq 0 \ \Longrightarrow u = 0\,, \\ u \succeq 0 \ \wedge \ v \succeq 0 \ \Longrightarrow u + v \succeq 0\,, \\ \lambda \geq 0 \ \wedge \ u \succeq 0 \ \Longrightarrow \lambda u \succeq 0\,, \end{aligned}$$

where, again, we have used the usual convention that $u \succeq v$ iff $v \preceq u$. The space F^1 or, more generally, the space F^N with the lexicographical ordering is an example of a linearly ordered vector space. (Given two vectors $\boldsymbol{u} = (u_i)_{i=1}^N$, $\boldsymbol{v} = (v_i)_{i=1}^N \in F^N$, recall that $\boldsymbol{u}$ is *lexicographically less than or equal to* $\boldsymbol{v}$, writing $\boldsymbol{u} \preceq \boldsymbol{v}$, iff, for some $i_0 \in \{1, \dots, N, N+1\}$, we have $u_i = v_i$ for $i = 1, \dots, i_0 - 1$ and $u_{i_0} < v_{i_0}$ if $i_0 \leq N$.) Considering a problem of linear programming, the space V will be the space of the "objective values" of a linear mapping $\gamma\colon W \to V$ whose value is to be maximized subject to the given constraints.

Let $A\colon W \to F^m$ be a linear mapping, let $\boldsymbol{b} \in F^m$ be a column vector, and let $\gamma\colon W \to V$ be a linear mapping. Then the *primal problem* of linear programming, which we consider, is to

$$\begin{aligned} &\text{maximize} \quad \gamma x \\ &\text{subject to} \quad Ax \leq \boldsymbol{b}\,. \end{aligned}$$

For example, when $F = \mathbb{R}$ and $W = \mathbb{R}^n$ and $V = \mathbb{R}^1$, then A corresponds to a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and γ corresponds to a row vector $\boldsymbol{c}^T \in \mathbb{R}^{1 \times n}$. Note that the case when $V = \mathbb{R}^N$ with the lexicographical ordering has some applications in the multiobjective optimization.

The symbol "ι" (Greek letter iota) transposes the next two elements; the elements are to be multiplied in the new order. For a vector $u \in V$ and a scalar $\lambda \in F$, we have

$$\iota u \lambda = \lambda u\,,$$

the λ-multiple of the vector u. If $\boldsymbol{u} = (u_i)_{i=1}^m \in V^m$ is an m-component column vector of vectors, then its transpose $\boldsymbol{u}^T$ is a row vector, which can be multiplied by the symbol ι from the left and by another column vector $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^m \in F^m$ of scalars from the right. We have

$$\iota \boldsymbol{u}^T \boldsymbol{\lambda} = \begin{pmatrix} \iota u_1 & \dots & \iota u_m \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} = \iota u_1 \lambda_1 + \dots + \iota u_m \lambda_m = \lambda_1 u_1 + \dots + \lambda_m u_m\,.$$

Note that, actually, the vector $u \in V$ induces a linear mapping

$$\begin{aligned} \iota u &\colon F \longrightarrow V\,, \\ \iota u &\colon \lambda \longmapsto \iota u \lambda = \lambda u \end{aligned}$$

for $\lambda \in F$. If $\alpha\colon W \to F$ is a linear form, then $\iota u \alpha\colon W \to V$ is the composition of the mappings. For an $x \in W$, we have

$$\iota u \alpha x = \iota u (\alpha x) = (\alpha x) u\,.$$

Analogously, the vector $\boldsymbol{u} \in V^m$ induces a linear mapping

$$\begin{aligned} \iota \boldsymbol{u}^T &\colon F^m \longrightarrow V\,, \\ \iota \boldsymbol{u}^T &\colon \ \boldsymbol{\lambda} \ \longmapsto \iota \boldsymbol{u}^T \boldsymbol{\lambda} \end{aligned}$$

for $\boldsymbol{\lambda} \in F^m$. If $A = (\alpha_i)_{i=1}^m\colon W \to F^m$ is a linear mapping, which is made up of m linear forms $\alpha_1, \dots, \alpha_m\colon W \to F$, then $\iota \boldsymbol{u}^T A\colon W \to V$ is the composition of the mappings. For an $x \in W$, we have

$$\iota \boldsymbol{u}^T A x = \iota \boldsymbol{u}^T (Ax) = \iota u_1 (\alpha_1 x) + \dots + \iota u_m (\alpha_m x) = (\alpha_1 x) u_1 + \dots + (\alpha_m x) u_m\,.$$

Conventions analogous to those above also apply when $u \in F$ or $\boldsymbol{u} \in F^m$.

Finally, the symbol o shall denote the zero linear form $o\colon W \to F$ on W with $ox = 0$ for all $x \in W$. The symbol $\boldsymbol{o}$ shall denote a column vector of zeros of the field F or the vector space V; the meaning will always be clear from the context. Inequalities between column vectors – like $Ax \leq \boldsymbol{b}$, $\boldsymbol{y} \leq \boldsymbol{b}$, $Ax \leq \boldsymbol{o}$, $\boldsymbol{\lambda} \geq \boldsymbol{o}$ or $\boldsymbol{u} \succeq \boldsymbol{o}$ – are understood componentwise.

2 ALGEBRAIC LINEAR PROGRAMMING

Let V be a linearly ordered vector space over a linearly ordered (possibly skew) field F, let W be a vector space over the field F, let $A\colon W \to F^m$ and $\gamma\colon W \to V$ be linear mappings, and let $\boldsymbol{b} \in F^m$ be a column vector. The following three results – Farkas' Lemma 1, Gale's Theorem 2 of the alternative, and Duality Theorem 3 for linear programming – were proved by Bartl (2007):

Lemma 1 (Farkas' Lemma)**.** *It holds*

$$\forall x \in W\colon\ Ax \leq \boldsymbol{o} \implies \gamma x \preceq 0 \tag{1}$$

if and only if

$$\exists \boldsymbol{u} \in V^m,\ \boldsymbol{u} \succeq \boldsymbol{o}\colon\ \iota\boldsymbol{u}^T A = \gamma\,. \tag{2}$$

Remark 1. Formula (2) essentially means that, given linear mappings $A\colon W \to F^m$ and $\gamma\colon W \to V$, there exists a non-negative linear mapping $\iota\boldsymbol{u}^T\colon F^m \to V$ which makes the following diagram commute:

$$\begin{tikzcd} W \arrow[r, "A"] \arrow[d, "\gamma"'] & F^m \arrow[dl, dashed, "\iota\boldsymbol{u}^T"] \\ V & \end{tikzcd}$$

We say that a linear mapping $L\colon F^m \to V$ is *non-negative* iff it preserves also the ordering, i.e., for all $\boldsymbol{\lambda} \in F^m$, if $\boldsymbol{\lambda} \geq \boldsymbol{o}$, then $L\boldsymbol{\lambda} \succeq 0$. Shorter algebraic proofs of Farkas' Lemma can be found in Bartl (2008, 2012a, and 2012b).

Theorem 2 (Gale's Theorem)**.** *It holds that*

$$\nexists x \in W\colon\ Ax \leq \boldsymbol{b} \tag{3}$$

if and only if

$$\exists \boldsymbol{\lambda} \in F^m,\ \boldsymbol{\lambda} \geq \boldsymbol{o}\colon\ \iota\boldsymbol{\lambda}^T A = o \,\wedge\, \iota\boldsymbol{\lambda}^T \boldsymbol{b} < 0\,. \tag{4}$$

Remark 2. Formula (3) says that the system of linear inequalities $Ax \leq \boldsymbol{b}$ has no solution. Formula (4) means in words that, in the space F^m, there exists a hyperplane that separates the subspace $\operatorname{Rng} A = \{\, Ax : x \in W \,\}$ from the shifted cone $\{\, \boldsymbol{y} \in F^m : \boldsymbol{y} \leq \boldsymbol{b} \,\}$. Indeed: The system $Ax \leq \boldsymbol{b}$ has no solution if and only if the set $\{\, \boldsymbol{y} \in F^m : \boldsymbol{y} \leq \boldsymbol{b} \,\}$ does not intersect the range $\operatorname{Rng} A$ of the mapping A. The column vector $\boldsymbol{\lambda}$ induces a linear form $\iota\boldsymbol{\lambda}^T\colon F^m \to F$ with $\iota\boldsymbol{\lambda}^T\colon \boldsymbol{y} \to \iota\boldsymbol{\lambda}^T\boldsymbol{y}$ for $\boldsymbol{y} \in F^m$. The equality $\iota\boldsymbol{\lambda}^T A = o$ means in words that the linear form is zero on the subspace $\operatorname{Rng} A$, i.e., the range $\operatorname{Rng} A$ is contained in the kernel of the form. Observe that $\iota\boldsymbol{\lambda}^T\boldsymbol{y} \leq \iota\boldsymbol{\lambda}^T\boldsymbol{b}$ for all $\boldsymbol{y} \in \{\, \boldsymbol{y} \in F^m : \boldsymbol{y} \leq \boldsymbol{b} \,\}$ if and only if $\boldsymbol{\lambda} \geq 0$. We can see hence that, choosing a constant $c \in F$ so that $\iota\lambda^T\boldsymbol{b} < c < 0$, the hyperplane $\{\, \boldsymbol{y} \in F^m : \iota\boldsymbol{\lambda}^T\boldsymbol{y} = c \,\}$ separates the subspace $\operatorname{Rng} A$ from the shifted cone $\{\, \boldsymbol{y} \in F^m : \boldsymbol{y} \leq \boldsymbol{b} \,\}$. See also Bartl (2012c).

Theorem 3 (Duality Theorem)**.** *Consider the following primal and dual problem of linear programming:*

$$\begin{array}{lll} (\mathrm{P}) & \text{maximize} & \gamma x \\ & \text{s.t.} & Ax \leq \boldsymbol{b}\,, \end{array} \qquad\qquad \begin{array}{lll} (\mathrm{D}) & \text{minimize} & \iota\boldsymbol{u}^T\boldsymbol{b} \\ & \text{s.t.} & \iota\boldsymbol{u}^T A = \gamma\,, \\ & & \boldsymbol{u} \succeq \boldsymbol{o}\,, \end{array}$$

where $x \in W$ and $\boldsymbol{u} \in V^m$ are variables. Then:

I. *If $x^* \in W$ is an optimal solution to the primal problem* (P)*, then there is an optimal solution $\boldsymbol{u}^* \in V^m$ to the dual problem* (D) *with $\gamma x^* = \iota\boldsymbol{u}^{*T}\boldsymbol{b}$.*

II. *If $\boldsymbol{u}^* \in V^m$ is an optimal solution to the dual problem* (D) *and the vector space V is non-trivial, then there is an optimal solution $x^* \in W$ to the primal problem* (P) *with $\gamma x^* = \iota\boldsymbol{u}^{*T}\boldsymbol{b}$.*

Remark 3. Farkas' Lemma 1 essential to prove Part I and Gale's Theorem 2 is necessary to prove Part II of Duality Theorem 3 (Bartl, 2007).

We have recalled three general results (Bartl, 2007). When we put $F = \mathbb{R}$, the field of the real numbers, take $W = \mathbb{R}^n$, a space of a finite dimension, and $V = \mathbb{R}^1$, the real axis, Farkas' Lemma 1, Gale's Theorem 2, and Duality Theorem 3, then we obtain the classical version of Farkas' Lemma (Farkas, 1902), Gale's Theorem of the alternative (Fan, 1956; Gale, 1960), and Duality Theorem for linear programming (Gale et al., 1951), respectively. See Bartl (2007, 2008, 2012a, and 2012b) for a more detailed discussion. Recall that the case when $V = \mathbb{R}^N$ with the lexicographical ordering has some applications in the multiobjective optimization.

3 INFINITE ALGEBRAIC LINEAR PROGRAMMING

3.1 Motivation

In the preceding section, we recalled some results with a finite system of linear inequalities $Ax \leq \boldsymbol{o}$ (Farkas' Lemma 1) or $Ax \leq \boldsymbol{b}$ (Gale's Theorem 2 and Duality Theorem 3). It has been an interesting question whether it is possible to obtain analogous generalized results with an infinite system of linear inequalities $Ax \leq \boldsymbol{o}$ or $Ax \leq \boldsymbol{b}$.

Farkas' Lemma 1 is a cornerstone in the theory due to Bartl (2007): all the results (Gale's Theorem 2, other theorems of the alternative, and Duality Theorem 3) follow from it. That is why we shall deal with an infinite version of Farkas' Lemma first. Thus, let M be an infinite index set and let $\alpha_i\colon W \to F$, for $i \in M$, be linear forms. We consider the infinite system of linear inequalities $Ax \leq \boldsymbol{o}$, or $\alpha_i x \leq 0$ for $i \in M$. Assuming that $\gamma x \preceq 0$ for all $x \in W$ such that $Ax \leq \boldsymbol{o}$, we should have

$$\gamma = \iota\boldsymbol{u}^T A = \sum_{i \in M} \iota u_i \alpha_i$$

for some non-negative $\boldsymbol{u} \in V^M$, i.e. some non-negative vectors $u_i \in V$ for $i \in M$. That is, for an $x \in W$, we should have

$$\gamma x = \iota \boldsymbol{u}^T A x = \sum_{i \in M} \iota u_i \alpha_i x \,.$$

However, the sum $\sum_{i \in M} \iota u_i \alpha_i x$ must be correct – we do not consider any additional concept such as topology or convergence here – whence, only a finite number of the terms can be non-zero. In addition, as in Remark 1, we should have the commutative diagram

$$\begin{tikzcd} W \arrow[r, "A"] \arrow[d, "\gamma"] & ? \arrow[dl, dashed, "\iota \boldsymbol{u}^T"] \\ V & \end{tikzcd}$$

perhaps with $\boldsymbol{u} \in V^M$, meaning that possibly all of the u_i can be non-zero or positive. Hence, we can guess that we should have $? = F^{(M)}$, the space of all infinite sequences with only a finite number of non-zero entries.

3.2 Definitions, counterexamples, and the constraint qualification

Let V be a linearly ordered vector space over a linearly ordered (possibly skew) field F and let W be a vector space over the field F.

Let M be a (finite or infinite) index set. Formally, a column vector $\boldsymbol{u} = (u_i)_{i \in M} \in V^M$ of vectors of the space V is a sequence or mapping

$$\begin{aligned} \boldsymbol{u}\colon M &\longrightarrow V, \\ \boldsymbol{u}\colon\ i &\longmapsto u_i \,. \end{aligned}$$

Now, for a set X, we write $\operatorname{Fin} X$ iff the set X is finite. Analogously then, a column vector $\boldsymbol{\lambda} = (\lambda_i)_{i \in M} \in F^{(M)}$ of scalars of the field F with a finite number of non-zero entries is a sequence of mapping

$$\begin{aligned} \boldsymbol{\lambda}\colon M &\longrightarrow F, \\ \boldsymbol{\lambda}\colon\ i &\longmapsto \lambda_i \end{aligned}$$

with

$$\operatorname{Fin}\{\, i \in M : \lambda_i \neq 0 \,\} \,.$$

To conclude, we have the two spaces

$$\begin{aligned} V^M &= \{\, \boldsymbol{u}\colon M \to V \,\}\,, \\ F^{(M)} &= \{\, \boldsymbol{\lambda}\colon M \to F : \operatorname{Fin}\{\, i \in M : \lambda_i \neq 0 \,\} \,\}\,, \end{aligned}$$

where M is an index set. Now, we conjecture that the following version of Farkas' Lemma could hold:

Hypothesis 1 (An infinite version of Farkas' Lemma). *Let V be a linearly ordered vector space over a linearly ordered (possibly skew) field F, let W be a vector space over the field F, let M be an index set, and let $A\colon W \to F^{(M)}$ and $\gamma\colon W \to V$ be linear mappings. Then*

$$\forall x \in W\colon\ Ax \leq \boldsymbol{o} \implies \gamma x \preceq 0$$

if and only if

$$\exists \boldsymbol{u} \in V^M,\, \boldsymbol{u} \succeq \boldsymbol{o}\colon\ \iota \boldsymbol{u}^T A = \gamma \,.$$

Indeed, although the "if" part of Hypothesis 1 is trivial, the "only if" part does not hold in general.

Counterexample 1. For simplicity, let us consider $F = \mathbb{R}$, the field of the real numbers, and $V = \mathbb{R}^1$, the one-dimensional real axis. Let $W = c_{00} = \mathbb{R}^{(\mathbb{N})}$ be the functional space of all sequences $\boldsymbol{x} = (x_i)_{i=1}^\infty$ of real numbers with only a finite number of non-zero entries.

Let $M = \mathbb{N} \cup \{\omega\} = \{1, 2, 3, \dots\} \cup \{\omega\}$ be the set of all finite natural numbers with a transfinite element. Let us consider the forms $\alpha_i \boldsymbol{x} = x_i$ for $i = 1, 2, 3, \dots$, and $\alpha_\omega \boldsymbol{x} = \sum_{i=1}^\infty -x_i$, putting $\gamma \boldsymbol{x} = \sum_{i=1}^\infty -i x_i$ for an $\boldsymbol{x} = (x_i)_{i=1}^\infty \in W$. In a less formal way, we can represent $\alpha_1, \alpha_2, \alpha_3, \dots$, and α_ω with γ as row vectors:

$$\begin{array}{rl} \alpha_1 &= (\ \ 1 \quad\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad \dots\ )\,, \\ \alpha_2 &= (\ \ 0 \quad\ \ 1 \quad\ \ 0 \quad\ \ 0 \quad \dots\ )\,, \\ \alpha_3 &= (\ \ 0 \quad\ \ 0 \quad\ \ 1 \quad\ \ 0 \quad \dots\ )\,, \\ \alpha_4 &= (\ \ 0 \quad\ \ 0 \quad\ \ 0 \quad\ \ 1 \quad \dots\ )\,, \\ \multicolumn{2}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ \alpha_\omega &= (\ -1 \quad -1 \quad -1 \quad -1 \quad \dots\ )\,, \\ \hline \gamma &= (\ -1 \quad -2 \quad -3 \quad -4 \quad \dots\ )\,. \end{array}$$

Thanks to the choice of the space $W = c_{00}$, only a finite number of the linear forms α_i is non-zero at a point $\boldsymbol{x} \in W$, and the form α_ω with the mapping γ are well defined because only a finite number of the terms is non-zero in the sums.

Now, choose an $\boldsymbol{x} = (x_i)_{i=1}^{\infty} \in W = c_{00}$. If $\alpha_i \boldsymbol{x} = x_i \le 0$ for $i = 1, 2, 3, \ldots$, and $\alpha_\omega \boldsymbol{x} \le 0$, i.e., $\sum_{i=1}^{\infty} x_i \ge 0$, then $\boldsymbol{x} = \boldsymbol{o}$, hence $\gamma \boldsymbol{x} = 0$, so $\gamma \boldsymbol{x} \preceq 0$. However, there exist no non-negative numbers $u_1, u_2, u_3, \ldots$, and u_ω such that $\gamma = u_\omega \alpha_\omega + \sum_{i=1}^{\infty} u_i \alpha_i$.

The counterexample motivates us to introduce a certain constraint qualification: we shall exclude the case described in Counterexample 1.

Definition 1 (F-linear independence)**.** Let $A\colon W \to F^{(M)}$ be a linear mapping so that we have an indexed collection $\{\alpha_i\}_{i \in M}$ of linear forms such that, for any $x \in W$, the set $\{\, i \in M : \alpha_i x \neq 0 \,\}$ is finite. Now, let $M^* \subseteq M$ be any subset of the index set M. We say that the subcollection $\{\alpha_i\}_{i \in M^*}$ is *F-linearly independent* iff

$$\forall \boldsymbol{\lambda}_{M^*} \in F^{M^*}\colon\ \iota \boldsymbol{\lambda}_{M^*}^T A_{M^*} = \sum_{i \in M^*} \iota \lambda_i \alpha_i = o \implies \boldsymbol{\lambda}_{M^*} = \boldsymbol{o}\,.$$

Definition 2 (Constrain Qualification (CQ))**.** Let $A\colon W \to F^{(M)}$ be a linear mapping. We say that the linear mapping A satisfies the constraint qualification (CQ) iff, for any subset $M^* \subseteq M$ such that the subcollection $\{\alpha_i\}_{i \in M^*}$ is F-linearly independent, and for any infinite subset $M^- \subseteq M^*$, there exists a point $x \in W$ such that

$$Ax \le \boldsymbol{o}\,, \qquad A_{M^-} x \neq \boldsymbol{o} \qquad \text{and} \qquad A_{M^* \setminus M^-} x = \boldsymbol{o}\,.$$

The latter two conditions mean that $\alpha_i x \neq 0$, hence $\alpha_i x < 0$, for at least one $i \in M^-$ and that $\alpha_i x = 0$ for all $i \in M^* \setminus M^-$.

Assuming the constraint qualification (CQ), the "only if" part of Farkas' Lemma (Hypothesis 1) becomes to hold true (see the next subsection).

Now, we shall be concerned with an infinite version of Gale's Theorem. Let us consider an infinite system $Ax \le \boldsymbol{b}$ with the linear mapping $A\colon W \to F^{(M)}$. Thus, it might seem plausible that we should have $\boldsymbol{b} \in F^{(M)}$. Then, however, the system $Ax \le \boldsymbol{b}$ would not be interesting: we would have a finite system $\alpha_i x \le b_i$ for $i \in M$ with $b_i \neq 0$ and the remaining, possibly infinite, part $\alpha_i x \le 0$ for $i \in M$ with $b_i = 0$. Therefore, we shall consider the more general case when $\boldsymbol{b} \in F^M$. Formally, a column vector $\boldsymbol{b} = (b_i)_{i \in M} \in F^M$ of scalars of the field F is a sequence or mapping

$$\begin{aligned} \boldsymbol{b}&\colon M \longrightarrow F\,, \\ \boldsymbol{b}&\colon\ i \ \longmapsto b_i\,. \end{aligned}$$

Naturally, we have to assume that the column vector $\boldsymbol{b}$ comprises only a finite number of negative entries. (Otherwise, the system $Ax \le \boldsymbol{b}$ could not have a solution as only a finite number of entries of the left-hand column can be non-zero.) In order that the sum $\iota \boldsymbol{\lambda}^T \boldsymbol{b} = \sum_{i \in M} \iota \lambda_i b_i$ is well defined, we shall require that only a finite number of the terms is non-zero, i.e., the set $\{\, i \in M : \lambda_i \neq 0 \ \wedge\ b_i \neq 0 \,\}$ is finite. Thus, we conjecture that the following version of Gale's Theorem could hold:

Hypothesis 2 (An infinite version of Gale's Theorem)**.** *Let W be a vector space over a linearly ordered (possibly skew) field F, let M be an index set, let $A\colon W \to F^{(M)}$ be a linear mapping, and let $\boldsymbol{b} \in F^M$ be a column vector. Under the assumption $\mathrm{Fin}\{\, i \in M : b_i < 0 \,\}$, it holds that*

$$\nexists x \in W\colon\ \ Ax \le \boldsymbol{b}$$

if and only if

$$\exists \boldsymbol{\lambda} \in F^M,\ \boldsymbol{\lambda} \ge \boldsymbol{o},\ \mathrm{Fin}\{\, i \in M : \lambda_i \neq 0 \ \wedge\ b_i \neq 0 \,\}\colon\ \ \iota \boldsymbol{\lambda}^T A = o \ \wedge\ \iota \boldsymbol{\lambda}^T \boldsymbol{b} < 0\,.$$

Again, while the "if" part of Hypothesis 2 is obvious, its "only if" part does not hold in general.

Counterexample 2. Take $F = \mathbb{R}$, the field of the real numbers, with $W = c_{00} = \mathbb{R}^{(\mathbb{N})}$, the functional space of all sequences $\boldsymbol{x} = (x_i)_{i=1}^{\infty}$ of real numbers with only a finite number of non-zero entries. Consider the system

$$\begin{array}{rl} -x_1 & \le -1\,, \\ -x_2 + x_1 & \le \tfrac{1}{2}\,, \\ -x_3 + x_2 & \le \tfrac{1}{4}\,, \\ -x_4 + x_3 & \le \tfrac{1}{8}\,, \\ -x_5 + x_4 & \le \tfrac{1}{16}\,, \\ \multicolumn{2}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \end{array}$$

Obviously, the system has no solution in the space $W = c_{00}$. However, no finite linear combination of the left-hand sides yields the zero linear form on W: all the left hand-sides have to be summed up; then, however, the sum of the right-hand sides is zero, not negative.

3.3 The main results

Let V be a linearly ordered vector space over a linearly ordered (possibly skew) field F, let W be a vector space over the field F, let M be an index set, let $A\colon W \to F^{(M)}$ be a linear mapping satisfying the constraint qualification (CQ), let $\boldsymbol{b} \in F^M$ be a column vector with $\operatorname{Fin}\{\, i \in M : b_i < 0 \,\}$, and let $\gamma\colon W \to V$ be a linear mapping. Then, the following three results hold true:

Lemma 4 (Farkas' Lemma)**.** *If*

$$\forall x \in W\colon\; Ax \leq \boldsymbol{o} \implies \gamma x \preceq 0\,, \tag{5}$$

then

$$\exists \boldsymbol{u} \in V^M,\; \boldsymbol{u} \succeq \boldsymbol{o}\colon\; \iota \boldsymbol{u}^T A = \gamma\,. \tag{6}$$

Theorem 5 (Gale's Theorem)**.** *If*

$$\nexists x \in W\colon\; Ax \leq \boldsymbol{b}\,, \tag{7}$$

then

$$\exists \boldsymbol{\lambda} \in F^{(M)},\; \boldsymbol{\lambda} \geq \boldsymbol{o}\colon\; \iota \boldsymbol{\lambda}^T A = o \;\wedge\; \iota \boldsymbol{\lambda}^T \boldsymbol{b} < 0\,. \tag{8}$$

Theorem 6 (Duality Theorem)**.** *Consider the following primal and dual problem of linear programming:*

$$\begin{array}{llll} (\mathrm{P}) & \text{maximize} & \gamma x & \\ & \text{s.t.} & Ax \leq \boldsymbol{b}\,, & \end{array} \qquad\qquad \begin{array}{lll} (\mathrm{D}) & \text{minimize} & \iota \boldsymbol{u}^T \boldsymbol{b} \\ & \text{s.t.} & \iota \boldsymbol{u}^T A = \gamma\,, \\ & & \boldsymbol{u} \succeq \boldsymbol{o}\,, \\ & \multicolumn{2}{l}{\operatorname{Fin}\{\, i \in M : u_i \neq 0 \;\wedge\; b_i \neq 0 \,\}\,,} \end{array}$$

where $x \in W$ and $\boldsymbol{u} \in V^M$ are variables. Then:

I. *If $x^* \in W$ is an optimal solution to the primal problem (P), then there is an optimal solution $\boldsymbol{u}^* \in V^M$ to the dual problem (D) with $\gamma x^* = \iota {\boldsymbol{u}^*}^T \boldsymbol{b}$.*

II. *If $\boldsymbol{u}^* \in V^M$ is an optimal solution to the dual problem (D) and the vector space V is non-trivial, then there is an optimal solution $x^* \in W$ to the primal problem (P) with $\gamma x^* = \iota {\boldsymbol{u}^*}^T \boldsymbol{b}$.*

Gale's Theorem 5 is surprising: if the system $Ax \leq \boldsymbol{b}$ has no solution, then, by (8), some finite subsystem of it has no solution. The condition $\operatorname{Fin}\{\, i \in M : \lambda_i \neq 0 \;\wedge\; b_i \neq 0 \,\}$ is not necessary in Gale's Theorem 5 (though conjectured in Hypothesis 2), but its variant is essential in the dual problem (D) in Duality Theorem 6. Let us observe that, if the set M is finite, e.g., $M = \{1, \ldots, m\}$, then the constraint qualification (CQ) is naturally satisfied – there is no infinite subset $M^- \subseteq M^* \subseteq M$. Thus, Farkas' Lemma 4, Gale's Theorem 5, and Duality Theorem 6 generalizes Farkas' Lemma 1, Gale's Theorem 2, and Duality Theorem 3, respectively. The proofs of the main results are rather long. That is why they are to be published elsewhere. Now, the results are quite general and abstract. It is an interesting question whether they can be applied to some problems of infinite linear programming whose solution is already known (e.g., Anderson & Nash, 1987), perhaps establishing a new approach to solving those problems. That question, however, is left to further research.

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