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Conformal and Holomorphic Mappings of Once-holed Tori

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ABSTRACT: We survey recent developments on conformal and holomorphic mappings of once-holed tori into Riemann surfaces of positive genus. We are mainly concerned with the sets of marked once-holed tori which allow handle-preserving conformal and holomorphic mappings into a given marked Riemann surface.

1 INTRODUCTION

By a *once-holed torus* we mean a noncompact Riemann surface of genus one with exactly one (Kerékjártó-Stoïlow) boundary component. For example, the Riemann surface obtained from a compact Riemann surface of genus one, or a *torus*, by removing one point is a once-holed torus, which will be called a *once-punctured torus*. Though all once-holed tori are homeomorphic to one another, no once-punctured torus is conformally equivalent to a once-holed torus which is not a once-punctured torus.

By the general uniformization theorem every Riemann surface of genus zero is conformally equivalent to a plane domain. Thus the core of the theory of Riemann surfaces should be occupied by studies of Riemann surfaces of positive genus. Once-holed tori are the simplest among the Riemann surfaces of positive genus, and every Riemann surface of positive genus includes a once-holed torus as a subdomain. Once-holed tori should play a role in the study of Riemann surfaces similar to that played by open disks for function theory on plane domains.

A topological condition on a holomorphic mapping of a Riemann surface into another sometimes places strong restrictions on the mapping or the Riemann surfaces. In this paper we are concerned with handle-preserving holomorphic mappings of once-holed tori into Riemann surfaces of positive genus. To be more precise we make some definitions.

A Riemann surface R of positive genus has one or more handles. A handle of R is specified by an ordered pair $\{a, b\}$ of simple loops a and b on R whose intersection number $a \times b$ is equal to one. Such a pair is called a *mark of handle* of R. A *marked Riemann surface* Y means a pair (R, χ), where R is a Riemann surface of positive genus and $\chi = \{a, b\}$ is a mark of handle of R.

Let $Y' = (R', \chi')$, $\chi' = \{a', b'\}$, be another marked Riemann surface. If a holomorphic mapping $f : R \to R'$ maps a and b onto loops freely homotopic to a' and b' on R', respectively, then we say that f is a holomorphic mapping of Y into Y' and use the notation $f : Y \to Y'$; we consider that f preserves the handles of R and R' specified by χ and χ'. If, in addition, $f : R \to R'$ is injective, then $f : Y \to Y'$ is said to be conformal.

Let $\mathfrak{T}$ be the set of marked once-holed tori, where two marked once-holed tori are identified if there is a conformal mapping of one onto the other. For a marked Riemann surface Y denote by $\mathfrak{T}_a[Y]$ (resp. $\mathfrak{T}_c[Y]$) the family of marked once-holed tori X such that there is a holomorphic (resp. conformal) mapping of X into Y. We are concerned with the sets $\mathfrak{T}_a[Y]$ and $\mathfrak{T}_c[Y]$, and survey recent developments. Extremal lengths will play a central role.

2 MARKED TORI

2.1 Moduli Disks

Bochner pointed out that every Riemann surface of finite genus can be conformally embedded into a compact Riemann surface of the same genus (Bochner, 1928). He applied the general uniformization theorem. Shiba introduced conformal embeddings called hydrodynamic continuations (Shiba, 1984; Shiba & Shibata, 1987), and showed their extremal properties (Shiba, 1987; Shiba, 1988; Shiba, 1989). Hydrodynamic continuations are a generalization of extremal parallel slit mappings on Riemann surfaces of genus zero to the case of higher genus. In this section we apply the results to once-holed tori.

Let $\mathbb{H}$ denote the upper half plane: $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. For each $\tau \in \mathbb{H}$ let G_τ be the additive group generated by 1 and τ, and set $T_\tau = \mathbb{C}/G_\tau$. We equip T_τ with conformal structure so that the natural projection $\pi_\tau : \mathbb{C} \to T_\tau$ is holomorphic. Then T_τ is a torus, that is, a compact Riemann surface of genus one. In general, let $[z_1, z_2]$ stands for the segment joining z_1 and z_2. The projection π_τ maps $[0, 1]$ and $[0, \tau]$ onto simple loops a_τ and b_τ on T_τ, respectively, which make a mark χ_τ of handle of T_τ. The correspondence $\tau \mapsto X_\tau := (T_\tau, \chi_\tau)$ defines a bijection of $\mathbb{H}$ onto the space of marked tori. The *modulus* of a marked torus is, by definition, the point of $\mathbb{H}$ corresponding to the marked torus.

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Let X be a marked once-holed torus. There is a conformal mapping of X into a marked torus by Bochner's theorem. However, there may be two or more marked tori into which X can be conformally embedded. Let $\Delta(X)$ denote the set of $\tau \in \mathbb{H}$ for which there is a conformal mapping of X into X_τ.

Theorem 1 (Shiba, 1987)**.** *The set $\Delta(X)$ is a closed disk or a point in $\mathbb{H}$.*

The disk $\Delta(X)$ will be referred to as the *moduli disk* of X. Let τ^* and ρ be the center and radius of $\Delta(X)$, respectively, where a singleton is considered as a closed disk of radius 0. If $\rho = 0$, then X is a marked once-punctured torus, and is conformally embedded into X_{τ^*} uniquely up to conformal automorphisms of X_{τ^*}. If $\rho > 0$, then X is not a marked once-punctured torus. Conformal mappings of X into the marked tori corresponding to the boundary points of $\Delta(X)$ are of special character. Each point τ on the boundary $\partial\Delta(X)$ is expressed as $\tau = \tau^* - i\rho e^{i\pi t}$ for some $t \in \mathbb{R}$. Then X is conformally embedded into X_τ in a unique manner up to conformal automorphisms of X_τ. Moreover, if $\iota_\tau : X \to X_\tau$ is conformal, then $\iota_\tau(X) = X_\tau \setminus \pi_\tau(\gamma)$ for some line segment γ of positive length with inclination $\pi t/2$ (Shiba, 1987). In particular, if τ is the bottom point of $\Delta(X)$, that is, if τ minimizes the imaginary part in $\Delta(X)$, then $t = 0$ and hence $\iota_\tau(X)$ is a horizontal slit domain of X_τ.

For any interior point τ of $\Delta(X)$ there are $\tau_0, \tau_1 \in \partial\Delta(X)$ such that the segment $[\tau_0, \tau_1]$ is a diameter of $\Delta(X)$ and that $\tau = (1-s)\tau_0 + s\tau_1$ for some $s \in (0,1)$. Let π be a holomorphic universal covering map of the unit disk $\mathbb{D}$ onto X. Each ι_{τ_j} induces a holomorphic mapping $\tilde{\iota}_{\tau_j} : \mathbb{D} \to \mathbb{C}$ such that $\pi_{\tau_j} \circ \tilde{\iota}_{\tau_j} = \iota_{\tau_j} \circ \pi$. The convex combination $\tilde{\iota}_\tau := (1-s)\tilde{\iota}_{\tau_0} + s\tilde{\iota}_{\tau_1}$ induces a conformal mapping $\iota_\tau : X \to X_\tau$ with $\pi_\tau \circ \tilde{\iota}_\tau = \iota_\tau \circ \pi$; if $\tau = \tau^*$, then τ_0 and τ_1 are not uniquely determined, but ι_τ does not depend on the choice of τ_0 and τ_1. The complement $X_\tau \setminus \iota_\tau(X)$ is the closure of a convex domain on X_τ. The mapping ι_τ possesses an interesting extremal property (see Theorem 3 below).

Let $\mathfrak{D}$ be the set of closed disks in $\mathbb{H}$; recall that a singleton is regarded as a closed disk of radius 0. Then the correspondence $\Delta : X \mapsto \Delta(X)$ is a mapping of the space $\mathfrak{T}$ of marked once-holed tori into $\mathfrak{D}$.

Theorem 2 (Masumoto, 1995)**.** *The mapping $\Delta : \mathfrak{T} \to \mathfrak{D}$ is bijective.*

Thus two marked once-holed are identical if and only if their moduli disks coincides with each other. Furthermore, for any closed disk D in $\mathbb{H}$ there is a marked once-holed torus whose moduli disk is exactly D. Note that if X is a marked once-punctured torus, then $\Delta(X)$ is a singleton, and vice versa.

2.2 Area Theorems

Let X be a marked once-holed torus. For each $\tau \in \Delta(X)$ let $\mathcal{C}_\tau(X)$ denote the set of conformal embeddings of X into X_τ. The natural projection $\pi_\tau : \mathbb{C} \to X_\tau$ induces a flat metric on X_τ. For $\iota \in \mathcal{C}_\tau(X)$ the area $\operatorname{Area}\bigl(X_\tau \setminus \iota(X)\bigr)$ of $X_\tau \setminus \iota(X)$ with respect to the metric will be denoted by $A(\iota)$. Set $A_\tau(X) = \sup A\bigl(\mathcal{C}_\tau(X)\bigr)$.

Theorem 3 (Shiba, 1993)**.** *Let τ^* and ρ be the center and radius of $\Delta(X)$, respectively. Suppose that $\rho > 0$. Then*

$$A_\tau(X) = \frac{\rho^2 - r^2}{2\rho}$$

for each $\tau \in \Delta(X)$, where $r = |\tau - \tau^*|$. The conformal mapping $\iota_\tau : X \to X_\tau$ attains the supremum $A_\tau(X)$.

In fact, any conformal mapping $\iota : X \to X_\tau$ that maximizes $A(\iota)$ in $\mathcal{C}_\tau(X)$ differs from ι_τ by a conformal automorphism of X_τ. Note that the function $\tau \mapsto A_\tau(X)$ on $\Delta(X)$ attains its maximum at the center τ^*.

Let R be a Riemann surface of genus zero, and fix a holomorphic local coordinate z around $p_0 \in R$ with $z(p_0) = 0$. Let $\mathcal{C}(R)$ be the family of conformal mappings ι of R into the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ such that

$$\iota(z) = \frac{1}{z} + a_1 z + a_2 z^2 + \cdots$$

near p_0. We denote the coefficient a_1 by $a(\iota)$, and set $\Delta(R) = \{a(\iota) \mid \iota \in \mathcal{C}(R)\}$. A theorem due to Grötzsch claims that $\Delta(R)$ is a closed disk in $\mathbb{C}$. Its boundary $\partial\Delta(R)$ is described by the first coefficients of extremal parallel slit mappings on R (Grötzsch, 1932). Schiffer showed that the average of the extremal horizontal and vertical slit mappings is again a conformal mapping in $\mathcal{C}(R)$, which maximizes the area of the complement of the image (Schiffer, 1943). Thus Theorem 1 and part of Theorem 3 are analogues of the classical results. To each conformal mapping of a marked once-holed torus X into a marked torus there corresponds an integrable holomorphic differential on X. This fact enables us to apply classical methods to the study of conformal mappings into tori.

The moduli disk $\Delta(X)$ of a marked once-holed torus X is also a disk with respect to the hyperbolic metric on $\mathbb{H}$, the complete conformal metric with curvature -1. If we choose the ratio $B(\iota) := A(\iota)/\operatorname{Area}(X_\tau) = \operatorname{Area}\bigl(X_\tau \setminus \iota(X)\bigr)/\operatorname{Area}(X_\tau)$ of areas instead of $A(\iota)$, then we obtain the following striking result. Set $B_\tau(X) = \sup B\bigl(\mathcal{C}_\tau(X)\bigr)$.

Theorem 4 (Shiba, 1993)**.** *Let τ_h^* and ρ_h be the hyperbolic center and radius of $\Delta(X)$, respectively. Suppose that $\rho_h > 0$. Then*

$$B_\tau(X) = \frac{\cosh\rho_h - \cosh r_h}{\sinh\rho_h}$$

for each $\tau \in \Delta(X)$, where r_h is the hyperbolic distance between τ and τ_h^*. The conformal mapping $\iota_\tau : X \to X_\tau$ attains the supremum $B_\tau(X)$.

Observe that $\tau \mapsto A_\tau(X)$ is the reciprocal of the density of the hyperbolic metric on the interior of $\Delta(X)$ while $\tau \mapsto \operatorname{Area}(X_\tau) = \operatorname{Im}\tau$ is the reciprocal of the density of the hyperbolic metric on $\mathbb{H}$. Thus $\tau \mapsto B_\tau(X) = A_\tau(X)/\operatorname{Area}(X_\tau)$ is the quotient of hyperbolic densities. We can apply this fact to derive Theorem 4 from Theorem 3 (Masumoto & Shiba, 1995).

For any marked once-holed torus X we can construct a family $\{X_s\}_{0 \leqq s \leqq 1}$ of marked once-holed tori with $X_0 = X$ such that the moduli disk $\Delta(X_s)$ shrinks continuously and tends to a point as $s \to 1$. This enables us to determine the possible values of complementary areas: $A\big(\mathcal{C}_\tau(X)\big) = \big[0, A_\tau(X)\big]$ and $B\big(\mathcal{C}_\tau(X)\big) = \big[0, B_\tau(X)\big]$ for all $\tau \in \Delta(X)$. It follows that for each interior point τ of $\Delta(X)$ there are uncountably many conformal mappings of X into X_τ (Masumoto, 1994).

2.3 Extremal Lengths

Let Γ be a family of curves on a Riemann surface R. For a linear density $\rho = \rho(z)\,|dz|$, $z = x + iy$, on R set

$$L_\rho(\Gamma) = \inf_{\gamma \in \Gamma'} \int_\gamma \rho(z)\,|dz| \quad \text{and} \quad A_\rho = \iint_R |\rho(z)|^2\,dx\,dy,$$

where Γ' is the subfamily consisting of locally rectifiable curves in Γ. The *extremal length* of Γ is, by definition, $\sup_\rho L_\rho(\Gamma)^2/A_\rho$, where the supremum is taken over all linear densities ρ on R such that $L_\rho(\Gamma)$ and A_ρ are not simultaneously 0 or $+\infty$. Extremal lengths are invariant under conformal mappings, that is, if $f : R \to R'$ is conformal, then the extremal length of the image family $f(\Gamma)$ is equal to that of Γ. Also, the extremal length is decreasing, that is, if a curve family Γ_1 includes another curve family Γ_2, then the extremal length of Γ_1 does not exceed that of Γ_2.

Let $X = (T, \chi)$, where $\chi = \{a, b\}$, be a marked once-holed torus. The extremal length $\lambda(X)$ of the free homotopy class of a is called the *basic extremal length* of X. It is equal to the extremal length of the singular homology class represented by a. If we set $\dot a = b$ and $\dot b = a^{-1}$, then we obtain another mark $\dot\chi = \{\dot a, \dot b\}$ of handle of T. Also, if $\ddot a$ is a simple loop freely homotopic to ab^{-1} and $\ddot b$ is identical with a, then $\ddot\chi = \{\ddot a, \ddot b\}$ is again a mark of handle of T. Setting $\dot X = (T, \dot\chi)$ and $\ddot X = (T, \ddot\chi)$, we define a point $\Lambda(X)$ of $\mathbb{R}_+^3$ by $\Lambda(X) = \big(\lambda(X), \lambda(\dot X), \lambda(\ddot X)\big)$, where $\mathbb{R}_+$ denotes the set of nonnegative real numbers: $\mathbb{R}_+ = [0, +\infty)$.

In general, for $A > 0$ set

$$U(A) = \left\{ z \in \mathbb{H} \;\middle|\; \operatorname{Im} z > \frac{1}{A} \right\}, \quad \dot U(A) = \left\{ z \in \mathbb{H} \;\middle|\; \operatorname{Im}\left(-\frac{1}{z}\right) > \frac{1}{A} \right\}, \quad \ddot U(A) = \left\{ z \in \mathbb{H} \;\middle|\; \operatorname{Im} \frac{1}{1-z} > \frac{1}{A} \right\}.$$

The set $U(A)$ is a half plane while $\dot U(A)$ and $\ddot U(A)$ are open disks of radius $A/2$ tangent to the real line at 0 and 1, respectively. If τ is the bottom point of the moduli disk $\Delta(X)$ of a marked once-holed torus X, then it is easy to verify that the basic extremal length $\lambda(X)$ is exactly $1/\operatorname{Im}\tau$ since X is realized as a horizontal slit domain of X_τ. This observation yields the following characterization of $\Delta(X)$ in terms of $\Lambda(X)$:

Theorem 5 (Masumoto, 1994)**.** *Let X be a marked once-holed torus, and set $U = U\big(\lambda(X)\big)$, $\dot U = \dot U\big(\dot\lambda(X)\big)$ and $\ddot U = \ddot U\big(\ddot\lambda(X)\big)$.*

(i) *If $\partial U \cap \partial \dot U \cap \partial \ddot U \neq \varnothing$, then $\Delta(X) = \partial U \cap \partial \dot U \cap \partial \ddot U$.*
(ii) *If $\partial U \cap \partial \dot U \cap \partial \ddot U = \varnothing$, then $U \cap \dot U \cap \ddot U$ is a nonempty circular triangle and $\Delta(X)$ is its inscribed disk.*

The hyperbolic diameter of $\Delta(X)$ is equal to $\log\big\{-Q\big(\Lambda(X)\big)/4\big\}$, where Q is the quadratic form of $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ defined by

$$Q(\boldsymbol{\xi}) = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1).$$

(Masumoto, 1994). It thus follows from Theorems 2 and 5 that Λ is an injection of $\mathfrak{T}$ into $\mathfrak{L} := \{\boldsymbol{\xi} \in \mathbb{R}_+^3 \mid Q(\boldsymbol{\xi}) + 4 \leqq 0\}$. In fact, we have the following theorem.

Theorem 6 (Masumoto, 1995)**.** *The mapping $\Lambda : \mathfrak{T} \to \mathfrak{L}$ is bijective.*

The eigenvalues of the coefficient matrix of Q is -1 and 2. The eigenspaces V_{-1} and V_2 corresponding to the eigenvalues -1 and 2 are the line $\xi_1 = \xi_2 = \xi_3$ and the plane $\xi_1 + \xi_2 + \xi_3 = 0$, respectively. The boundary $\partial\mathfrak{L}$ of $\mathfrak{L}$ in $\mathbb{R}_+^3$ is a component of a hyperboloid of two sheets. Note that X is a marked once-punctured torus if and only if $\Lambda(X) \in \partial\mathfrak{L}$.

We use Λ to make $\mathfrak{T}$ a 3-dimensional real analytic manifold with boundary. As a set, $\partial\mathfrak{T}$ is the Teichmüller space of a once-punctured torus and its complement $\mathfrak{T} \setminus \partial\mathfrak{T}$ is the reduced Teichmüller space of a once-holed torus which is not a once-punctured torus. Each of these Teichmüller spaces carries a real analytic structure, which is compatible with the real analytic structure on $\mathfrak{T}$ introduced above (Masumoto, 1995).

2.4 Conformal and Holomorphic Mappings into Marked Tori

Let Y be a marked Riemann surface. Recall that $\mathfrak{T}_a[Y]$ (resp. $\mathfrak{T}_c[Y]$) denotes the family of marked once-holed tori X such that there is a holomorphic (resp. conformal) mapping of X into Y. Set $\mathfrak{S}_i[Y] = \Sigma(\mathfrak{T}_i[Y])$ and $\mathfrak{L}_i[Y] = \Lambda(\mathfrak{T}_i[Y])$ for $i = a, c$. In this section we describe $\mathfrak{T}_i[Y]$ for marked tori $Y = \{R, \chi\}$, where $\chi = \{a, b\}$.

If τ is the modulus of Y, then Theorem 1 shows that $\mathfrak{T}_c[Y]$ consists of marked once-holed tori X whose moduli disks $\Delta(X)$ contain τ. We can also describe $\mathfrak{T}_c[Y]$ in terms of extremal lengths. As in the case of marked once-holed tori, let $\lambda(Y)$, $\dot{\lambda}(Y)$ and $\ddot{\lambda}(Y)$ denote the extremal lengths of the free homotopy classes of a , b and ab^{-1}, respectively. Set $\Lambda(Y) = \big(\lambda(Y), \lambda(\dot{Y}), \lambda(\ddot{Y})\big)$. If Y' is a marked once-punctured torus obtained from Y by removing one point, then $\Lambda(Y) = \Lambda(Y') \in \partial\mathfrak{L}$. In general, for $\boldsymbol{\xi} \in \mathcal{L}$ denote by $C(\boldsymbol{\xi})$ the set of $\boldsymbol{\eta} \in \mathbb{R}^3$ such that $Q(\boldsymbol{\eta} - \boldsymbol{\xi}) \leqq 0$ and $Q(\boldsymbol{\eta}) \leqq Q(\boldsymbol{\xi})$. It is a cone with vertex at $\boldsymbol{\xi}$ whose axis of symmetry is parallel to V_{-1}. Note that $C(\boldsymbol{\xi})$ is a subset of $\mathfrak{L}$. Then the next theorem follows at once from Theorem 5. We will improve it later (see Theorem 11 below).

Theorem 7. *If Y is a marked torus, then $\mathfrak{L}_c[Y] = C\big(\Lambda(Y)\big)$.*

Next we examine $\mathfrak{T}_a[Y]$. The following theorem reflects a special character of tori.

Theorem 8. *If Y is a marked torus, then $\mathfrak{T}_a[Y] = \mathfrak{T}$.*

This is an immediate consequence of the Behnke-Stein theorem, which assures the existence of holomorphic differentials with prescribed periods on noncompact Riemann surfaces. Shiba studied the existence problem of holomorphic mappings into tori with prescribed boundary behavior (Shiba, 1978; Shiba, 1980; Shiba, 1981).

3 MARKED ONCE-HOLED TORI

3.1 Ordering Relation

For $X, X' \in \mathfrak{T}$ we say that X is *smaller* than X' and write $X \preceq X'$ if there is a conformal mapping of X into X'. In other words, if X is conformally equivalent to some subdomain of X', then $X \preceq X'$. If this is the case, we also say that X' is *larger* than X and use the notation $X' \succeq X$.

The relation $\preceq$ is clearly reflexive and transitive. Since it is also anti-symmetric (Komatu & Mori, 1952), $(\mathfrak{T}, \preceq)$ is an ordered set. It is never totally ordered. For $Y \in \mathfrak{T}$ the set $\mathfrak{T}_c[Y]$ is thus the set of lower bounds of Y.

The ordering relation $\preceq$ is expressed in terms of moduli disks and extremal lengths. We have namely the following theorem.

Theorem 9 (Masumoto, 1995)**.** *For $X, X' \in \mathfrak{T}$ the following statements are equivalent to one another.*

(i) $X \preceq X'$.
(ii) $\Delta(X) \supset \Delta(X')$.
(iii) $Q\big(\Lambda(X) - \Lambda(X')\big) \leqq 0$ *and* $Q\big(\Lambda(X)\big) \leqq Q\big(\Lambda(X')\big)$.

We give an immediate consequence of Theorem 9. It is an improvement of Theorem 2.

Theorem 10. *The mapping Δ is a decreasing isomorphism of the ordered set $(\mathfrak{T}, \preceq)$ onto the ordered set $(\mathfrak{D}, \subset)$.*

3.2 Conformal and Holomorphic Mappings into Marked Once-holed Tori

We begin with another application of Theorem 9. The following theorem improves Theorem 7. Note that the set of marked tori is canonically identified with the set of marked once-punctured tori.

Theorem 11. *If Y is a marked once-holed torus, then the identity $\mathfrak{L}_c[Y] = C\big(\Lambda(Y)\big)$ holds.*

In general, for a once-holed torus T let $\mathcal{C}(T)$ denote the set of free homotopy classes of nondividing loops on T. For each $\Gamma \in \mathcal{C}(T)$ denote by $L_e(\Gamma)$ (resp. $L_h(\Gamma)$) the extremal length of Γ (resp. the length of the hyperbolic geodesic in Γ).

Now, let T_1 and T_2 be once-holed tori, and let κ be a homeomorphism of T_1 onto T_2. We then ask whether there is a conformal mapping of T_1 into T_2 homotopic to κ. The mapping κ induces a bijection $\kappa_* : \mathcal{C}(T_1) \to \mathcal{C}(T_2)$. If there exists a conformal mapping of T_1 into T_2 homotopic to κ, then we have

$$L_e\big(\kappa_*(\Gamma)\big) \leqq L_e(\Gamma) \quad \text{and} \quad L_h\big(\kappa_*(\Gamma)\big) \leqq L_h(\Gamma)$$

for all $\Gamma \in \mathcal{C}(T_1)$. For extremal length the converse is also valid:

Theorem 12 (Masumoto, 1997)**.** *If $L_e\big(\kappa_*(\Gamma)\big) \leqq L_e(\Gamma)$ holds for all $\Gamma \in \mathcal{C}(T_1)$, then there is a conformal mapping of T_1 into T_2 homotopic to κ.*

On the other hand, for hyperbolic length the converse does not hold. In fact, we have the following theorem:

Theorem 13 (Masumoto, 1997; Masumoto, 2012)**.** *For any once-holed torus T there is a once-holed torus T' together with a homeomorphism κ of T' onto T such that there is a holomorphic mapping of T' into T homotopic to κ but there are no conformal mappings of T' into T homotopic to κ.*

Therefore, we have no theorem similar to Theorem 12 for hyperbolic length since holomorphic mappings decrease hyperbolic lengths by the Schwarz-Pick theorem. It follows from Theorem 13 that $\mathfrak{T}_c[Y]$ is a proper subset of $\mathfrak{T}_a[Y]$ for any $Y \in \mathfrak{T}$.

Set $A(r) = \{z \in \mathbb{C} \mid 1 < |z| < r\}$ for $r \in (1, +\infty]$. For $r_1, r_2 \in (1, +\infty]$, there is a holomorphic mapping of $A(r_1)$ into $A(r_2)$ inducing a nontrivial homomorphisms between the fundamental groups of $A(r_j)$, $j = 1, 2$, if and only if $r_1 \leqq r_2$ (Schiffer, 1943; Huber, 1951; Huber, 1953; Jenkins, 1953; Landau & Osserman, 1959; Landau & Osserman, 1959/60; Marden et al., 1967). This theorem implies that for doubly connected Riemann surfaces R_1 and R_2 of finite moduli if there is a homotopically nontrivial holomorphic mapping of R_1 into R_2, then there is a homotopically nontrivial conformal mapping of R_1 into R_2. Theorem 13 shows that for once-holed tori the situation is completely different.

Let Y be a marked once-holed torus. Theorem 9 implies that $\lambda(X) \geqq \lambda(Y)$ for any $X \in \mathfrak{T}_c[Y]$. For $X \in \mathfrak{T}_a[Y]$ the same inequality does not hold any longer. However, $\lambda(X)$ cannot arbitrarily small as the following theorem shows:

Theorem 14 (Masumoto, 2009). *Let Y be a marked once-holed torus. Then the inequality*

$$\lambda(X) \geqq \frac{8\log(\sqrt{2}+1)}{\pi}\frac{\lambda(Y)}{5\lambda(Y)^2+4}$$

holds for all $X \in \mathfrak{T}_c[Y]$.

4 MARKED RIEMANN SURFACES

4.1 Handle Conditions

Studies on $\mathfrak{T}_a[Y]$ and $\mathfrak{T}_c[Y]$ for general marked Riemann surfaces Y are still in its infancy. In our previous work we examined $\mathfrak{T}_c[Y]$ (Masumoto, 2007; Masumoto, 2011). Among other things, we described the set of upper bounds of $\mathfrak{T}_c[Y]$ in terms of extremal length of free homotopy classes of simple loops on Y. Very recently, we have developed a new method for investigating the sets $\mathfrak{T}_a[Y]$ and $\mathfrak{T}_c[Y]$. In this section we summarize some of the results.

Let $\mathcal{P}(X)$ be a mathematical statement, where the free variable X ranges over $\mathfrak{T}$. Thus $\mathcal{P}(X)$ is an expression on marked once-holed tori X. It is required that the meaning of $\mathcal{P}(X)$ is so clear that for each $X \in \mathfrak{T}$ whether $\mathcal{P}(X)$ is true or false is determined without ambiguity. It is called a *handle condition* if it possesses the following property: For $X_1, X_2 \in \mathfrak{T}$, if $X_1 \preceq X_2$ and $\mathcal{P}(X_2)$ is true, then $\mathcal{P}(X_1)$ is also true.

For an arbitrary statement $\mathcal{P}(X)$ on $X \in \mathfrak{T}$ define a mapping $v[\mathcal{P}] : \mathfrak{T} \to \{0,1\}$ by

$$v[\mathcal{P}](X) = \begin{cases} 1 & \text{if } \mathcal{P}(X) \text{ is true,} \\ 0 & \text{if } \mathcal{P}(X) \text{ is false.} \end{cases}$$

Then $\mathcal{P}(X)$ is a handle condition if and only if $v[\mathcal{P}]$ is decreasing.

Example 1. Let Y be a fixed marked Riemann surface. The statement $\mathcal{P}_a(X)$ that there is a holomorphic mapping of X into Y is a handle condition. If we restrict ourselves to injective holomorphic mappings, we obtain another handle condition $\mathcal{P}_c(X)$. Thus $\mathcal{P}_c(X)$ states that there is a conformal mapping of X into Y.

For a handle condition $\mathcal{P}(X)$ we denote by $\mathfrak{T}[\mathcal{P}]$ the set of marked once-holed tori X such that $\mathcal{P}(X)$ is true, or $v[\mathcal{P}](X) = 1$. For the handle conditions $\mathcal{P}_a(X)$ and $\mathcal{P}_c(X)$ in Example 1 we have $\mathfrak{T}[\mathcal{P}_a] = \mathfrak{T}_a[Y]$ and $\mathfrak{T}[\mathcal{P}_c] = \mathfrak{T}_c[Y]$.

We give two theorems on the shape of $\mathfrak{T}[\mathcal{P}]$. The first one describes $\mathfrak{L}[\mathcal{P}] := \Lambda(\mathfrak{T}[\mathcal{P}])$. Recall that the eigenspaces of the coefficient matrix of the quadratic form Q are the line V_{-1} and the plane V_2. Let $\boldsymbol{e} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \in V_{-1}$. We define a function $e[\mathcal{P}]$ on V_2 as follows. For $\boldsymbol{\zeta} \in V_2$ let $e[\mathcal{P}](\boldsymbol{\zeta})$ be the infimum of $t \in \mathbb{R}$ such that $\boldsymbol{\zeta} + t\boldsymbol{e} \in \mathfrak{L}[\mathcal{P}]$; if no such t exists, then we set $e[\mathcal{P}](\boldsymbol{\zeta}) = +\infty$. Let $V_2[\mathcal{P}]$ be the set of $\boldsymbol{\zeta}$ with $e[\mathcal{P}](\boldsymbol{\zeta}) < +\infty$, and set

$$\mathcal{L}[\mathcal{P}] = \{\boldsymbol{\zeta} + t\boldsymbol{e} \mid \boldsymbol{\zeta} \in V_2[\mathcal{P}],\ t > e[\mathcal{P}](\boldsymbol{\zeta})\} \quad \text{and} \quad \bar{\mathcal{L}}[\mathcal{P}] = \{\boldsymbol{\zeta} + t\boldsymbol{e} \mid \boldsymbol{\zeta} \in V_2[\mathcal{P}],\ t \geqq e[\mathcal{P}](\boldsymbol{\zeta})\}.$$

Theorem 15 (Masumoto, 2012). *If $\mathfrak{L}[\mathcal{P}] \neq \varnothing$, then $V_2[\mathcal{P}]$ is identical with V_2, and $e[\mathcal{P}]$ is Lipschitz continuous on V_2. Furthermore, the inclusion relations*

$$\mathcal{L}[\mathcal{P}] \subset \mathfrak{L}[\mathcal{P}] \subset \bar{\mathcal{L}}[\mathcal{P}]$$

hold.

For the proof we apply Theorem 12. Note that $\mathcal{L}[\mathcal{P}]$ is a domain and that $\bar{\mathcal{L}}[\mathcal{P}]$ is its closure. Therefore, $\mathfrak{T}[\mathcal{P}]$ is a connected set. In general, $e[\mathcal{P}]$ is *not* differentiable.

To give the second theorem we introduce a new global coordinate system on $\mathfrak{T}$. Set $\mathfrak{S} = \mathbb{H} \times [0, 1)$. For $(\tau, l) \in \mathfrak{S}$ define a marked once-holed torus $T_\tau^{(l)}$ by $T_\tau^{(l)} = T_\tau \setminus \pi_\tau([0, l])$. It is a horizontal slit domain of the torus T_τ. Choose a mark $\chi_\tau^{(l)} = \{a_\tau^{(l)}, b_\tau^{(l)}\}$ of handle of $T_\tau^{(l)}$ so that the inclusion mapping $T_\tau^{(l)} \hookrightarrow T_\tau$ induces a conformal mapping of $X_\tau^{(l)} := (T_\tau^{(l)}, \chi_\tau^{(l)})$ into X_τ. Every marked once-holed torus is conformally equivalent to some $X_\tau^{(l)}$ (Shiba, 1984; Shiba & Shibata, 1987). In fact, the correspondence $(\tau, l) \mapsto X_\tau^{(l)}$ is a bijection of $\mathfrak{S}$ onto $\mathfrak{T}$ (Masumoto, 1995). Its inverse will be denoted by Σ. Thus $\Sigma(X_\tau^{(l)}) = (\tau, l)$. Note that the bottom point of the moduli disk of $X_\tau^{(l)}$ is exactly τ. The composite $\Lambda \circ \Sigma^{-1}$ is real-analytic on $\mathfrak{S}$ (Masumoto, 2012).

For $t \in [0, +\infty]$ denote by $\mathcal{H}(t)$ the set of $z \in \mathbb{H}$ with $0 < \operatorname{Im} z < t$ and by $\bar{\mathcal{H}}(t)$ its closure in $\mathbb{H}$. Note that $\mathcal{H}(0) = \bar{\mathcal{H}}(0) = \varnothing$ while $\mathcal{H}(+\infty) = \bar{\mathcal{H}}(+\infty) = \mathbb{H}$. Let $\Pi : \mathfrak{S} \to \mathbb{H}$, $\Pi(z, l) = z$, be the natural projection. For a handle condition $\mathcal{P}(X)$ set $\mathfrak{S}[\mathcal{P}] = \Sigma(\mathfrak{T}[\mathcal{P}])$ and $\mathbb{H}[\mathcal{P}] = \Pi(\mathfrak{S}[\mathcal{P}])$. In the proof of the next result Theorem 10 plays a fundamental role.

Theorem 16 (Masumoto, 2012). *For any handle condition $\mathcal{P}(X)$ there is a constant $\lambda[\mathcal{P}] \in [0, +\infty]$ such that*

$$\mathcal{H}\left(\frac{1}{\lambda[\mathcal{P}]}\right) \subset \mathbb{H}[\mathcal{P}] \subset \bar{\mathcal{H}}\left(\frac{1}{\lambda[\mathcal{P}]}\right),$$

where $1/0 = +\infty$ and $1/(+\infty) = 0$.

In other words, for $\tau \in \mathbb{H}$,

(i) if $\operatorname{Im} \tau > 1/\lambda[\mathcal{P}]$, then $\mathcal{P}(X_\tau^{(l)})$ is false for any l, while

(ii) if $\operatorname{Im} \tau < 1/\lambda[P]$, then $\mathcal{P}(X_\tau^{(l)})$ is true for some l.

We call the number $\lambda[\mathcal{P}]$ the *critical extremal length* for the handle condition $\mathcal{P}(X)$. If $\lambda[\mathcal{P}] = +\infty$, then $\mathcal{P}(X)$ is false for all $X \in \mathfrak{T}$, and vice versa.

4.2 Conformal and Holomorphic Mappings into Marked Riemann Surfaces

Let Y be a marked Riemann surface, and let $\mathcal{P}_a(X)$ and $\mathcal{P}_c(X)$ be the handle conditions in Example 1. Since $\mathfrak{T}_a[Y] = \mathfrak{T}[\mathcal{P}_a]$ and $\mathfrak{T}_c[Y] = \mathfrak{T}[\mathcal{P}_c]$, we can apply the results in the preceding section. In this section we give characteristic properties of the sets. We write $\mathcal{L}_i[Y] = \mathcal{L}[\mathcal{P}_i]$, $\lambda_i[Y] = \lambda[\mathcal{P}_i], \ldots$ for $i = a, c$.

Applying a normal family argument, we can show that $\mathfrak{T}_a[Y]$ is closed in $\mathfrak{T}$. We thus obtain the following theorem from Theorem 15:

Theorem 17 (Masumoto, 2012). $\mathfrak{L}_a[Y] = \bar{\mathcal{L}}_a[Y]$.

Therefore, $\mathfrak{T}_a[Y]$ is a closed domain. Moreover, it is a deformation retract of $\mathfrak{T}$. Note that $\mathfrak{L}_a[Y]$ satisfies the interior and exterior cone conditions.

On the other hand, $\mathbb{H}_a[Y]$ turns out to be open in $\mathbb{H}$. Hence the next theorem follows from Theorem 16. For the proof we perform surgery on horizontal slit tori.

Theorem 18 (Masumoto, 2012). $\mathbb{H}_a[Y] = \mathcal{H}_a[Y]$.

Since the behavior of extremal lengths under holomorphic mappings is uncontrollable, the existence of the extremal critical length $\lambda_a[Y]$ is quite remarkable. In contrast, $\mathbb{H}_c[Y]$ is not open if Y is a marked once-holed torus.

Finally, we compare $\lambda_a[Y]$ with $\lambda_c[Y]$. It follows from $\mathfrak{T}_a[Y] \supset \mathfrak{T}_c[Y] \neq \varnothing$ that $\lambda_a[Y] \leqq \lambda_c[Y] < +\infty$. In fact we can establish the following theorem. For the proof we again use the surgery on horizontal slit tori mentioned above.

Theorem 19 (Masumoto, 2012). $\lambda_a[Y] < \lambda_c[Y] < +\infty$.

If Y is not a marked torus, then there is a marked once-holed torus $\tilde{Y}$ such that $\mathfrak{T}_a[\tilde{Y}] = \mathfrak{T}_a[Y]$. It thus follows from Theorem 14 that $\lambda_a[Y]$ is positive. On the other hand, if Y is a marked torus, then $\lambda_a[Y] = 0$ by Theorem 8.

Let $A(r) = \{z \in \mathbb{C} \mid 1 < |z| < r\}$ for $r \in (1, +\infty]$. Fixing $r_0 \in (1, +\infty]$, denote by ρ_a (resp. ρ_c) the supremum of ρ for which there is a homotopically nontrivial holomorphic (resp. conformal) mapping of $A(\rho)$ into $A(r_0)$. It is trivial that $r_0 \leqq \rho_c \leqq \rho_a$. Schiffer's theorem shows that the identities $r_0 = \rho_c = \rho_a$ actually hold. Note that the extremal length of the loops in $A(r)$ separating the boundary components is exactly $2\pi/\log r$. Theorem 19 thus makes a sharp contrast with the consequence of Schiffer's theorem.

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