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Singularly Impulsive Dynamical Systems with Time Delay: Razumikhin Stability

Nataša A. Kablar [1], Vladimir Kvrgić[2], Mirko Bućan [3], and Dragutin Lj. Debeljković [4]
[1]N. A. Kablar is with Lola Institute and Faculty of Computer Science, 11000 Belgrade, Serbia.
[2]V. Kvrgić is with Lola Institute, 11000 Belgrade, Serbia.
[3]M. Bućan is with Lola Institute, 11000 Belgrade, Serbia.
[4]D. Lj. Debeljković is with Faculty of Mechanical Engineering, University of Belgrade, 11000 Belgrade, Serbia.

ABSTRACT: In this paper we introduce *new* class of system, so called *singularly impulsive* or *generalized impulsive dynamical systems with time delay*. Dynamics of this system is characterized by the set of differential and difference equations with time delay, and algebraic equations. They represent the class of hybrid systems, where algebraic equations represent constraints that differential and difference equations with time delay need to satisfy. In this paper we present model, assumptions on the model, and two classes of singularly impulsive dynamical systems with delay - time dependent and state dependent. Further, we present Lyapunov - Krasovskii and Razumikhin stability results for the class of singularly impulsive dynamical systems with time delay.

1 INTRODUCTION

Modern complex engineering systems as well as biological and physiological systems typically possess a multi-echelon hierarchical hybrid architecture characterized by continuous-time dynamics at the lower levels of hierarchy and discrete-time dynamics at the higher levels of the hierarchy. Hence, it is not surprising that hybrid systems have been the subject of intensive research over the past recent years (see Branicky et al. (1998), Ye et al. (1998 b), Haddad, Chellaboina and Kablar (2001a-b)). Such systems include dynamical switching systems Branicky (1998), Leonessa et al. (2000), nonsmooth impact and constrained mechanical systems, Back et al. (1993), Brogliato (1996), Brogliato et al. (1997), biological systems Lakshmikantham et al. (1989), demographic systems Liu (1994), sampled-data systems Hagiwara and Araki (1988), discrete-event systems Passino et al. (1994), intelligent vehicle/highway systems Lygeros et al. (1998) and flight control systems, etc. The mathematical descriptions of many of these systems can be characterized by impulsive differential equations, Simeonov and Bainov (1985), Liu (1988), Lakshmikantham et al. (1989, 1994), Bainov and Simeonov (1989, 1995), Kulev and Bainov (1989), Lakshmikantham and Liu (1989), Hu et al. (1989), Samoilenko and Perestyuk (1995), Haddad, Chellaboina and Kablar (2001a-b). Impulsive dynamical systems can be viewed as a subclass of hybrid systems.

Motivated by the results on impulsive dynamical systems presented in Haddad, Chellaboina, and Kablar (2001, 2005), the authors previous work on singular or generalized systems, and results on singularly impulsive dynamical systems published in Kablar(2003, 2010) we presented new class of *singularly impulsive* or *generalized impulsive dynamical systems with time delay*. It presents novel class of hybrid systems and generalization of impulsive dynamical systems to incorporate singular nature of the systems and time delays. Extensive applications of this class of systems can be found in contact problems and in hybrid systems.

We present mathematical model of the singularly impulsive dynamical systems with time delay. We show how it can be viewed as general systems from which impulsive dynamical systems with time delay, singular continuous-time systems with time delay and singular dicrete-time systems with time delay, as well as without time delay, follow. Then we present Assumptions needed for the model and the division of this class of systems to time-dependent and state-dependent singularly impulsive dynamical systems with time delay with respect to the resetting set.

In this paper for the class of nonlinear singularly impulsive dynamical systems with time delay we develop Lyapunov - Krasovskii and Razumikhin stability results. Results are further specialized to linear case. Note that for addressing the stability of the zero solution of a singularly impulsive dynamical system the usual stability definitions are valid. Finally, we draw some conclusions and define future work.

At first, we establish definitions and notations. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, let $\mathcal{N}$ denote the set of nonnegative integers, and let I_n or I denote the $n \times n$ identity matrix. Furthermore, let $\partial\mathcal{S}, \dot{\mathcal{S}}, \bar{\mathcal{S}}$ denote the boundary, the interior, and a closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. Finally, let C^0 denote the set of continuous functions and C^r denote the set of functions with r continuous derivatives.

2 MATHEMATICAL MODEL OF SINGULARLY IMPULSIVE DYNAMICAL MODEL WITH TIME DELAY

A singularly impulsive dynamical system with delay consists of three elements:

1. A possibly singular continuous-time dynamical equation with time delay, which governs the motion of the system between resetting events;

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2. A possibly singular difference equation with time delay, which governs the way the states are instantaneously changed when a resetting occurs; and

3. A criterion for determining when the states of the system are to be reset.

Mathematical model of these systems is described with

$$E_{\mathrm{c}}\dot{x}(t) = f_{\mathrm{c}}(x(t,\tau))+G_{\mathrm{c}}(x(t,\tau))u_{\mathrm{c}}(t),$$
$$(t, x(t,\tau), u_{\mathrm{c}}(t)) \notin \mathcal{S}, \tag{2.1}$$
$$E_{\mathrm{d}}\triangle x(t) = f_{\mathrm{d}}(x(t,\tau))+G_{\mathrm{d}}(x(t,\tau))u_{\mathrm{d}}(t),$$
$$(t, x(t,\tau), u_{\mathrm{c}}(t)) \in \mathcal{S}, \tag{2.2}$$
$$y_{\mathrm{c}}(t) = h_{\mathrm{c}}(x(t,\tau))+J_{\mathrm{c}}(x(t,\tau))u_{\mathrm{c}}(t),$$
$$(t, x(t,\tau), u_{\mathrm{c}}(t)) \notin \mathcal{S}, \tag{2.3}$$
$$y_{d}(t) = h_{\mathrm{d}}(x(t,\tau)+J_{\mathrm{d}}(x(t,\tau))u_{\mathrm{d}}(t),$$
$$(t, x(t,\tau), u_{\mathrm{c}}(t)) \in \mathcal{S}, \tag{2.4}$$

where $t \geq 0$, $\tau > 0$, $x(0) = x_0$, $x(t,\tau) \in \mathcal{D} \subset \mathbb{R}^{\ltimes} \times \mathbb{N}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u_{\mathrm{c}} \in \mathcal{U}_{\mathrm{c}} \subset \mathbb{R}^{\gtrdot_{\mathrm{c}}}$, $u_{\mathrm{d}}(t_k) \in \mathcal{U}_{\mathrm{d}} \subset \mathbb{R}^{\gtrdot_{\mathrm{d}}}$, t_k denotes k^{th} instant of time at which $(t, x(t,\tau), u_{\mathrm{c}}(t))$ intersects $\mathcal{S}$ for a particular trajectory $x(t,\tau)$ and input $u_{\mathrm{c}}(t), y_{\mathrm{c}}(t) \in \mathbb{R}^{\lessdot_{\mathrm{c}}}, y_{\mathrm{d}}(t_k) \in \mathbb{R}^{\lessdot_{\mathrm{d}}}$, $f_{\mathrm{c}} : \mathcal{D} \to \mathbb{R}^{\ltimes}$ is Lipschitz continuous and satisfies $f_{\mathrm{c}}(0) = 0$, $G_{\mathrm{c}} : \mathcal{D} \to \ltimes \times \gtrdot_{\mathrm{c}}$, $f_{\mathrm{d}} : \mathcal{D} \to \mathbb{R}^{\ltimes}$ is continuous and satisfies $f_{\mathrm{d}}(0) = 0$, $G_{\mathrm{d}} : \mathcal{D} \to \mathbb{R}^{n \times m_{\mathrm{d}}}$, $h_{\mathrm{c}} : \mathcal{D} \to \mathbb{R}^{l_{\mathrm{c}}}$ and satisfies $h_{\mathrm{c}}(0) = 0$, $J_{\mathrm{c}} : \mathcal{D} \to \mathbb{R}^{l_{\mathrm{c}} \times m_{\mathrm{c}}}$, $h_{\mathrm{d}} : \mathcal{D} \to \mathbb{R}^{l_{\mathrm{d}}}$ and satisfies $h_{\mathrm{d}}(0) = 0$, $J_{\mathrm{d}} : \mathcal{D} \to \mathbb{R}^{l_{\mathrm{d}} \times m_{\mathrm{d}}}$, and $\mathcal{S} \subset [0,\infty) \times \mathbb{R}^n \times \mathcal{U}_{\mathrm{c}}$ is the *resetting set*. Here, as in Haddad, Chellaboina, and Kablar (2001a) we assume that $u_{\mathrm{c}}(\cdot)$ and $u_{\mathrm{d}}(\cdot)$ are restricted to the class of *admissible* inputs consisting of measurable functions $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t)) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$ for all $t \geq 0$ and $k \in \mathcal{N}_{[0,t)} \equiv k : 0 \leq t_k < t$, where the constraint set $\mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$ is given with $(0,0) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$. We refer to the differential equation (2.1) as the *continuous-time dynamics with time delay*, and we refer to the difference equation (2.2) as the *resetting law*.

Matrices E_{c}, E_{d} may be singular matrices. In case $E_{\mathrm{c}} = I$, $E_{\mathrm{d}} = I$, and $\tau = 0$ (2.1)–(2.4) represent standard impulsive dynamical systems described in Haddad, Chellaboina, and Kablar (2001a), and Haddad, Kablar, and Chellaboina (2000, 2005), where stability, dissipativity, feedback interconnections, optimality, robustness, and disturbance rejection has been analyzed. In absence of discrete dynamics they specialize to singular continuous-time systems, with further specialization $E_{\mathrm{c}} = I$ to standard continuous-time systems. If only discrete dynamics is present they specialize to singular discrete-time systems, with further specialization $E_{\mathrm{d}} = I$ to standard discrete-time systems.

In case $E_{\mathrm{c}} = I$, $E_{\mathrm{d}} = I$, and $\tau \neq 0$, (2.1)–(2.4) represent standard impulsive dynamical systems with time delay. In absence of discrete dynamics they specialize to singular continuous-time systems with time delay, with further specialization $E_{\mathrm{c}} = I$ to standard continuous-time systems with time delay. If only discrete dynamics is present they specialize to singular discrete-time systems with time delay, with further specialization $E_{\mathrm{d}} = I$ to standard discrete-time systems with time delay.

Therefore, theory of the singularly impulsive or generalized impulsive dynamical systems with time delay once developed, can be viewed as a generalization of the singular and impulsive dynamical system with time delay theory, unifying them into more general new system theory.

In what follows is given basic setting and division of this class of systems with respect to the definition of the resetting sets, accompanied with adequate assumptions needed for the model.

We make the following additional assumptions:

A1. $(0, x_0, u_{\mathrm{c}0}) \notin \mathcal{S}$, where $x(0) = x_0$ and $u_{\mathrm{c}}(0) = u_{\mathrm{c}0}$, that is, the initial condition is not in $\mathcal{S}$.

A2. If $(t, x(t,\tau), u_{\mathrm{c}}(t)) \in \bar{\mathcal{S}} \backslash \mathcal{S}$ then there exists $\epsilon > 0$ such that, for all $0 < \delta < \epsilon$, $s(t+\delta; t, x(t,\tau), u_{\mathrm{c}}(t+\delta)) \notin \mathcal{S}$.

A3. If $(t_k, x(t_k), u_{\mathrm{c}}(t_k)) \in \partial S \cap \mathcal{S}$ then there exists $\epsilon > 0$ such that, for all $0 < \delta < \epsilon$ and $u_{\mathrm{d}}(t_k) \in \mathcal{U}_{\mathrm{d}}$, $s(t_k + \delta; t_k, E_{\mathrm{d}}x(t_k) + f_{\mathrm{d}}(x(t_k)) + G_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k), u_{\mathrm{c}}(t_k + \delta)) \notin \mathcal{S}$.

A4. We assume consistent initial conditions (and prior and after every resetting).

Assumption A1 ensures that the initial condition for the resetting differential equation (2.1), (2.2) is not a point of discontinuity, and this assumption is made for convenience. If $(0, x_0, u_{\mathrm{c}0}) \in \mathcal{S}$, then the system initially resets to $E_{\mathrm{d}}x_0^+ = E_{\mathrm{d}}x_0 + f_{\mathrm{d}}(x_0) + G_{\mathrm{d}}(x_0)u_{\mathrm{d}}(0)$ which serves as the initial condition for the continuous dynamics (2.1). It follows from A3 that the trajectory then leaves $\mathcal{S}$. We assume in A2 that if a trajectory reaches the closure of $\mathcal{S}$ at a point that does not belong to $\mathcal{S}$, then the trajectory must be directed away from $\mathcal{S}$, that is, a trajectory cannot enter $\mathcal{S}$ through a point that belongs to the closure of $\mathcal{S}$ but not to $\mathcal{S}$. Finally, A3 ensures that when a trajectory intersects the resetting set $\mathcal{S}$, it instantaneously exits $\mathcal{S}$, see Figure 1. We make the following remarks.

Figure 1. Resetting Set.

Remark 2.1. *It follows from A3 that resetting removes the pair $(t_k, x_k, u_{\mathrm{c}}(t_k))$ from the resetting set $\mathcal{S}$. Thus, immediately after resetting occurs, the continuous-time dynamics (2.1), and not the resetting law (2.2), becomes the active element of the singularly impulsive dynamical system.*

Remark 2.2. *It follows from A1-A3 that no trajectory can intersect the interior of $\mathcal{S}$. According to A1, the trajectory $x(t)$ begins outside the set $\mathcal{S}$. Furthermore, it follows from A2 that a trajectory can only reach $\mathcal{S}$ through a point belonging* *to both $\mathcal{S}$ and its boundary. Finally, from A3, it follows that if a trajectory reaches a point $\mathcal{S}$ that is on the boundary of $\mathcal{S}$, then the trajectory is instantaneously removed from $\mathcal{S}$. Since a continuous trajectory starting outside of $\mathcal{S}$ and intersecting the interior of $\mathcal{S}$ must first intersect the boundary of $\mathcal{S}$, it follows that no trajectory can reach the interior of $\mathcal{S}$.*

Remark 2.3. *It follows from A1-A3 and Remark 1.2 that $\partial\mathcal{S} \cup \mathcal{S}$ is closed and hence the resetting times t_k are well defined and distinct.*

Remark 2.4. *Since the resetting times are well defined and distinct, and since the solutions to (2.1) exist and are unique, it follows that the solutions of the singularly impulsive dynamical system (2.1), (2.2) also exist and are unique over a forward time interval.*

In Haddad, Chellaboina and Kablar (2001a), the resetting set $\mathcal{S}$ is defined in terms of a countable number of functions $n_k : \mathbb{R}^n \to (0, \infty)$, and is given by

$$\mathcal{S} = \cup_k \{(n_k(x), x, u_c(n_k(x)) : x \in \mathbb{R}^n\}. \tag{2.5}$$

The analysis of singularly impulsive dynamical systems with time delay and with a resetting set of the form (2.5) can be quite involved. In particular, such systems exhibit Zenoness, beating, as well as confluence phenomena wherein solutions exhibit infinitely many transitions in a finite times, and coincide after a given point of time, Haddad, Chellaboina and Kablar (2001a). In this paper we assume that existence and uniqueness properties of a given singularly impulsive dynamical system with time delay are satisfied in forward time. Furthermore, since singularly impulsive dynamical systems of the form (2.1)-(2.4) involve impulses at variable times they are time-varying systems.

Here we will consider singularly impulsive dynamical systems involving two distinct forms of the resetting set $\mathcal{S}$. In the first case, the resetting set is defined by a prescribed sequence of times which are independent of state x. These equations are thus called *time-dependent singularly impulsive dynamical systems with time delay*. In the second case, the resetting set is defined by a region in the state space that is independent of time. These equations are called *state-dependent singularly impulsive dynamical systems with time delay*.

2.1 Time-Dependent Singularly Impulsive Dynamical Systems with Time Delay

Time-dependent singularly impulsive dynamical systems with time delay can be written as (2.1)–(2.4) with $\mathcal{S}$ defined as

$$\mathcal{S} = n \times \mathbb{R}^n \times \mathcal{U}_c, \tag{2.6}$$

where

$$n = t_1, t_2, \ldots \tag{2.7}$$

and $0 < t_1 < t_2 < \ldots$ are prescribed resetting times. When an infinite number of resetting times are used and $t_k \to \infty$ as $k \to \infty$, then $\mathcal{S}$ is closed. Now (2.1)–(2.4) can be rewritten in the form of the *time-dependent singularly impulsive dynamical system with time delay*

$$E_c\dot{x}(t) = f_c(x(t,\tau)) + G_c(x(t,\tau))u_c(t),\ t \neq t_k, \tag{2.8}$$
$$E_d\triangle x(t) = f_d(x(t,\tau)) + G_d(x(t,\tau))u_d(t),\ t = t_k, \tag{2.9}$$
$$y_c(t) = h_c(x(t,\tau)) + J_c(x(t,\tau))u_c(t),\ t \neq t_k, \tag{2.10}$$
$$y_d(t) = h_d(x(t,\tau)) + J_d(x(t,\tau))u_d(t),\ t = t_k. \tag{2.11}$$

Since $0 \notin \tau$ and $t_k < t_{k+1}$, $\tau > 0$, it follows that the assumptions A1–A3 are satisfied. Since time-dependent singularly impulsive dynamical systems with time delay involve impulses at a fixed sequence of times, they are time-varying systems.

Remark 2.5. *Standard continuous-time and discrete-time dynamical systems as well as sampled-data systems can be treated as special cases of singularly impulsive dynamical systems. For details see [1].*

Remark 2.6. *The time-dependent singularly impulsive dynamical system with time delay (2.8)–(2.11), with $E_c = I$ and $E_d = I$ includes as a special case the impulsive control problem addressed in the literature wherein at least one of the state variables of the continuous-time plant can be changed instantaneously to any given value given by an impulsive control at a set of control instants τ, Haddad, Chellaboina and Kablar (2001a).*

2.2 State-Dependent Singularly Impulsive Dynamical Systems with Time Delay

State-dependent singularly impulsive dynamical systems with time delay can be written as (2.1)–(2.4) with $\mathcal{S}$ defined as

$$\mathcal{S} = [0, \infty) \times \mathcal{Z}, \tag{2.12}$$

where $\mathcal{Z} = \mathcal{Z}_x \times \mathcal{U}_c$ and $\mathcal{Z}_x \subset \mathbb{R}^n$. Therefore, (2.1)–(2.4) can be rewritten in the form of the *state-dependent singularly impulsive dynamical system with time delay*

$$E_c\dot{x}(t) = f_c(x(t,\tau)) + G_c(x(t,\tau))u_c(t), \quad (x(t,\tau), u_c(t)) \notin \mathcal{Z}, \tag{2.13}$$

$$E_d\triangle x(t) = f_d(x(t,\tau)) + G_d(x(t,\tau))u_d(t), \quad (x(t,\tau), u_c(t)) \in \mathcal{Z}, \tag{2.14}$$

$$y_c(t) = h_c(x(t,\tau)) + J_c(x(t,\tau))u_c(t), \quad (x(t,\tau), u_c(t)) \notin \mathcal{Z}, \tag{2.15}$$

$$y_d(t) = h_d(x(t,\tau)) + J_d(x(t,\tau))u_d(t), \quad (x(t,\tau), u_c(t)) \in \mathcal{Z}. \tag{2.16}$$

We assume that $(x_0, u_{c0}) \notin \mathcal{Z}$, $\tau > 0$, $(0,0) \notin \mathcal{Z}$, and that the resetting action removes the pair (x, u_c) from the set $\mathcal{Z}$; that is, if $(x, u_c) \in \mathcal{Z}$ then $(E_d x + f_d(x) + G_d(x)u_d, u_c) \notin \mathcal{Z}$, $u_d \in \mathcal{U}_d$. In addition, we assume that if at time t the trajectory $(x(t,\tau), u_c(t)) \in \bar{\mathcal{Z}}\backslash\mathcal{Z}$, then there exists $\epsilon > 0$ such that for $0 < \delta < \epsilon$, $(x(t+\tau+\delta), u_c(t+\delta)) \notin \mathcal{Z}$.

These assumptions represent the specialization of A1–A3 for the particular resetting set (2.12). It follows from these assumptions that for a particular initial condition, the resetting times $\tau_k(x_0)$ are distinct and well defined. Since the resetting set $\mathcal{Z}$ is a subset of the state space and is independent of time, state-dependent singularly impulsive dynamical systems with time delay are time-invariant systems. Finally, in the case where $\mathcal{S} \equiv [0,\infty) \times \mathbb{R}^n \times \mathcal{Z}_{u_c}$, where $\mathcal{Z}_{u_c} \subset \mathcal{U}_c$ we refer to (2.13)–(2.16) as an input-dependent singularly impulsive dynamical system with time delay. Both these cases represent a generalization to the impulsive control problem considered in the literature.

3 STABIILITY OF SINGULARLY IMPUSLIVE DYNAMICAL SYSTEMS WITH TIME DELAY: RAZUMIKHIN STABILITY

We consider linear singular time-invariant systems with single delay

$$E_c\dot{x}(t) = A_c x(t) + A_{c1}x(t-\tau), \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{3.17}$$

$$E_d x(k+1) = A_d x(k) + A_{d1}x(k-\tau), \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{3.18}$$

where $x \in \mathbb{R}^n$, A_c, A_d, A_{c1}, A_{d1} are given $n \times n$ real matrices, and E_c, E_d may be singular matrices. The usual initial condition is in the form of

$$x_0(t) = \phi(t), \qquad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{3.19}$$

for $t, k \in [-\tau, 0]$, where ϕ is a continuous function.

We will use Razumikhin Theorem and Lyapunov-Krasovskii Stability Theorem to discuss the stability of the system. We will restrict ourselves to using the bounded quadratic Lyapunov function or Lyapunov-Krasovski functional, and aim at arriving at stability criteria that can be written in the form of Linear Matrix Inequalities (LMI) or a closely related form. Efficient numerical methods are available to solve LMI's.

We state here two important results we will need in the rest of the paper.

Since we use the bounded quadratic Lyapunov function, we only need to use the following restricted form of the Razumikhin Theorem given for singularly impulsive dynamical systems.

Proposition 3.1. *A singularly impulsive dynamical system with time-delay, with maximum time delay τ, is asymptotically stable if there exists a bounded quadratic Lyapunov function $V(x)$ such that for some $\epsilon > 0$, it satisfies* Lyapunov function condition

$$V(x(t)) \geq \epsilon\|E_c x(t)\|^2, \qquad (t, x(t,\tau)) \notin \mathcal{Z} \tag{3.20}$$

$$V(x(k)) \geq \epsilon\|E_d x(k)\|^2, \qquad (t, x(t,\tau)) \in \mathcal{Z} \tag{3.21}$$

and its derivative along the system trajectory $\dot{V}(x(t))$ satisfies Razumikhin derivative condition

$$\dot{V}(x(t)) \leq -\epsilon\|E_c x(t)\|^2, \qquad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{3.22}$$

$$\Delta V(x(k)) \leq -\epsilon\|E_d x(k)\|^2, \qquad (t, x(t,\tau)) \in \mathcal{Z}, \tag{3.23}$$

whenever

$$V(x(t+\xi)) \leq pV(x(t)), \quad -\tau \leq \xi \leq 0, \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{3.24}$$

$$V(x(k+\xi)) \leq pV(x(k)), \quad -\tau \leq \xi \leq 0, \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{3.25}$$

for some constant $p > 1$.

Similarly, we can state a restricted version of Lyapunov-Krasovskii Stability Theorem developed for the class of singularly impulsive dynamical systems.

Proposition 3.2. *A singularly impulsive dynamical system with time-delay system is asymptotically stable if there exists a bounded quadratic Lyapunov-Krasovskii functional $V(\phi)$ such that for some $\epsilon > 0$, it satisfies* Lyapunov-Krasovskii functional condition

$$V(\phi(t)) \geq \epsilon \|E_{\mathrm{c}}\phi(0)\|^2, \qquad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{3.26}$$
$$V(\phi(k)) \geq \epsilon \|E_{\mathrm{d}}\phi(0)\|^2, \qquad (t, x(t,\tau)) \in \mathcal{Z}, \tag{3.27}$$

and its derivative along the system trajectory,

$$\dot{V}(\phi(t)) = \dot{V}(x(t))|_{x(t)=\phi(t)},\ (t, x(t,\tau)) \notin \mathcal{Z}, \tag{3.28}$$
$$\Delta V(\phi(k)) = \Delta V(x(k))|_{x(k)=\phi(k)},\ (t, x(t,\tau)) \in \mathcal{Z}, \tag{3.29}$$

satisfies Lyapunov-Krasovskii derivative condition

$$\dot{V}(x(t)) \leq -\epsilon \|E_{\mathrm{c}}\phi(0)\|^2, \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{3.30}$$
$$\delta V(x(k)) \leq -\epsilon \|E_{\mathrm{d}}\phi(k_0)\|^2, \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{3.31}$$

4 DELAY-INDEPENDENT STABILITY CRITERIA BASED ON RAZUMIKHIN THEOREM

4.1 Systems with single delay

Consider the system with single delay described by (3.17)–(3.18). We will use the Razumikhin Theorem to obtain a simple stability condition using the Lyapunov function

$$V(x(t)) = x(t)^{\mathrm{T}} E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} x(t),\ (t, x(t,\tau)) \notin \mathcal{Z} \tag{4.32}$$
$$V(x(k)) = x(k)^{\mathrm{T}} E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{c}} x(k),\ (t, x(t,\tau)) \in \mathcal{Z}. \tag{4.33}$$

Proposition 4.1. *The system described by (3.17)-(3.18) is asymptotically stable if there exist a scalar $\alpha > 0$, and a real symmetric matrix P such that*

$$\begin{bmatrix} A_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}} P A_{\mathrm{c}} + \alpha p E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} & E_{\mathrm{c}}^{\mathrm{T}} P A_{\mathrm{c1}} \\ A_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} & \alpha E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} \end{bmatrix} < 0, \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.34}$$

and

$$\begin{bmatrix} A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}} - E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} + \alpha p E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} & A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d1}} \\ A_{\mathrm{d1}}^{\mathrm{T}} P A_{\mathrm{d}} & \alpha E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} + A_{\mathrm{d1}}^{\mathrm{T}} P A_{\mathrm{d1}} \end{bmatrix} < 0, \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{4.35}$$

Proof. We will use Proposition (3.1). Choose Lyapunov function V as in (4.32)–(4.33). Since (4.34)–(4.35) implies $E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} > 0$, and $E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} > 0$ we can conclude that for some sufficiently small $\epsilon > 0$, the Lyapunov function condition

$$V(x(t)) \geq \epsilon \|E_{\mathrm{c}} x(t)\|^2, \qquad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.36}$$
$$V(x(k)) \geq \epsilon \|E_{\mathrm{d}} x(t)\|^2, \qquad (t, x(t,\tau)) \in \mathcal{Z}, \tag{4.37}$$

is satisfied. Now consider the derivative of $V(x)$ along the trajectory of the system 3.17–3.18,

$$\dot{V}(x(t)) = \frac{d}{dt} V(x(t)) = 2x(t)^{\mathrm{T}}(t) E_{\mathrm{c}}^{\mathrm{T}} P [A_{\mathrm{c}} x(t) + A_{\mathrm{c1}} x(t-\tau)]\ (t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.38}$$

$$\begin{aligned} \Delta V(x(k)) = {} & V(x(k+1)) - V(x(k)) = \\ & x(k)^{\mathrm{T}}(t) A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}} x(k) \\ & + x(k)^{\mathrm{T}} A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d1}} x(k-\tau) \\ & + x(k-\tau)^{\mathrm{T}} A_{\mathrm{d1}}^{\mathrm{T}} P A_{\mathrm{d}} x(k) \\ & + x(k-\tau)^{\mathrm{T}} A_{\mathrm{d1}}^{\mathrm{T}} P A_{\mathrm{d1}} x(k-\tau) \\ & - x(k)^{\mathrm{T}} E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} x(k) \\ & (t, x(t,\tau)) \in \mathcal{Z}, \end{aligned} \tag{4.39}$$

Whenever x satisfies

$$V(x(t+\theta)) < pV(x(t)) \text{ for all } -\tau \le \theta \le 0, \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.40}$$

$$V(x(k+\theta)) < pV(x(k)) \text{ for all } -\tau \le \theta \le 0, \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{4.41}$$

for some $p > 1$, we can conclude that for any $\alpha > 0$

$$\begin{aligned} \dot{V}(x(t)) \quad &\le \quad 2x^{\mathrm{T}}(t)E_{\mathrm{c}}^{\mathrm{T}}P[A_{\mathrm{c}}x(t) + A_{\mathrm{c1}}x(t-\tau)] \\ &+\alpha[px^{\mathrm{T}}(t)E_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}}x(t) - x^{\mathrm{T}}(t-\tau)E_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}}x(t-\tau)] \\ &= \phi_0^{\mathrm{T}}\begin{bmatrix} A_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{c}} + \alpha pE_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} & E_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{c1}} \\ A_{\mathrm{c1}}^{\mathrm{T}}PE_{\mathrm{c}} & \alpha E_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} + A_{\mathrm{d1}}^{\mathrm{T}}PA_{\mathrm{d1}} \end{bmatrix}\phi_0, \\ &(t, x(t,\tau)) \notin \mathcal{Z}, \end{aligned} \tag{4.42}$$

$$\begin{aligned} &\Delta V(x(k)) \le \\ &x(k)^{\mathrm{T}}(t)A_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}}x(k) + x(k)^{\mathrm{T}}A_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d1}}x(k-\tau) \\ &+x(k-\tau)^{\mathrm{T}}A_{\mathrm{d1}}^{\mathrm{T}}PA_{\mathrm{d}}x(k) + x(k-\tau)^{\mathrm{T}}A_{\mathrm{d1}}^{\mathrm{T}}PA_{\mathrm{d1}}x(k-\tau) \\ &+\alpha[px^{\mathrm{T}}(k)E_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{d}}x(k) - x^{\mathrm{T}}(k-\tau)E_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{d}}x(k-\tau)] \\ &= \phi_0^{\mathrm{T}}\begin{bmatrix} A_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} - E_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{d}} + \alpha pE_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{d}} & A_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d1}} \\ A_{\mathrm{d1}}^{\mathrm{T}}PA_{\mathrm{d}} & \alpha E_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{d}} + +A_{\mathrm{d1}}^{\mathrm{T}}PA_{\mathrm{d1}} \end{bmatrix}\phi_{k_0}, \\ &\quad (t, x(t,\tau)) \in \mathcal{Z}, \end{aligned} \tag{4.43}$$

where

$$\phi_0 = \begin{pmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-\tau) \end{pmatrix}^{\mathrm{T}}, \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.44}$$

$$\phi_{k_0} = \begin{pmatrix} x^{\mathrm{T}}(k) & x^{\mathrm{T}}(k-\tau) \end{pmatrix}^{\mathrm{T}}, \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{4.45}$$

The inequality (4.34)–(4.35) implies that for some sufficiently small $\delta > 0$, $p = 1 + \delta$

$$\begin{bmatrix} A_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{c}} + \alpha pE_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} & E_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{c1}} \\ A_{\mathrm{c1}}^{\mathrm{T}}PE_{\mathrm{c}} & \alpha E_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} \end{bmatrix} < 0, \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.46}$$

$$\begin{bmatrix} A_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} - E_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{d}} + \alpha pE_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{d}} & A_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d1}} \\ A_{\mathrm{d1}}^{\mathrm{T}}PA_{\mathrm{d}} & \alpha E_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{d}} + +A_{\mathrm{d1}}^{\mathrm{T}}PA_{\mathrm{d1}} \end{bmatrix} < 0, \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{4.47}$$

which, according to (4.42)–(4.43), implies that the Razumikhin derivative condition

$$\dot{V}(x(t)) \le -\epsilon\|E_{\mathrm{c}}x(t)\|^2, \qquad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.48}$$

$$\Delta V(x(k)) \le -\epsilon\|E_{\mathrm{d}}x(k)\|^2, \qquad (t, x(t,\tau)) \in \mathcal{Z}, \tag{4.49}$$

is satisfied. According to Proposition (3.1), singularly impulsive system (3.17)–(3.18) is asymptotically stable. □

4.2 Systems with distributed delay

Now consider singularly impulsive dynamical system with distributed delays

$$E_{\mathrm{c}}\dot{x}(t) = A_{\mathrm{c}}x(t) + \int_{\tau}^{0} A_{\mathrm{c1}}x(t+\theta)\mathrm{d}\theta, \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.50}$$

$$E_{\mathrm{d}}x(k+1) = A_{\mathrm{d}}x(k) + \sum_{\tau}^{0} A_{\mathrm{d1}}x(k+\theta), \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{4.51}$$

where $A_{\mathrm{c}}, A_{\mathrm{d}}$ are given constant matrices and $A_{\mathrm{c1}}, A_{\mathrm{d1}}$ are given matrix valued function of $\theta \in [\tau, 0]$. We can again use Lyapunov function $V(x) = x^{\mathrm{T}}E_{\mathrm{c,d}}^{\mathrm{T}}PE_{\mathrm{c,d}}x$ to study the stability of the system and conclude the following.

Proposition 4.2. *The system with distributed delays described by (4.50)–(4.51) is asymptotically stable if there exist symmetric matrix P, and scalar function*

$$\alpha(\theta) > 0, for \leq \theta \leq \tau, \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.52}$$
$$\alpha(\theta) > 0, for \leq \theta \leq \tau, \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{4.53}$$

and a symmetric matrix functions $R_{\mathrm{c}}\theta)$ and $R_{\mathrm{d}}\theta)$ such that

$$A_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}} P A_{\mathrm{c}} + \textstyle\int_{-\tau}^{0} R_{\mathrm{c}}(\theta)\mathrm{d}\theta < 0, \tag{4.54}$$
$$A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}} + E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} + \textstyle\sum_{-\tau}^{0} R_{\mathrm{d}}(\theta) < 0, \tag{4.55}$$
$$\begin{pmatrix} p\alpha E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} - R_{\mathrm{c}}(\theta) & E_{\mathrm{c}}^{\mathrm{T}} P A_{\mathrm{c}1}(\theta) \\ A_{\mathrm{c}1}(\theta)^{\mathrm{T}} P E_{\mathrm{c}} & -\alpha E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} \end{pmatrix} < 0$$
$$\text{for } 0 \leq \theta \leq \tau. \tag{4.56}$$
$$\begin{pmatrix} p\alpha E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} - R_{\mathrm{d}}(\theta) & A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}1}(\theta) \\ A_{\mathrm{d}1}(\theta)^{\mathrm{T}} P A_{\mathrm{d}} & -\alpha E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} + + A_{\mathrm{d}1}^{\mathrm{T}} P A_{\mathrm{d}1} \end{pmatrix} < 0$$
$$\text{for } 0 \leq \theta \leq \tau. \tag{4.57}$$

Proof. Use the Razumikhin Theorem in a similar way to the proof of Proposition **??**. Since (4.56)–(4.57) implies $E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} > 0$ and $E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} > 0$, we can conclude that the Lyapunov function $V(x) = x^{\mathrm{T}} E_{\mathrm{c,d}}^{\mathrm{T}} P E_{\mathrm{c,d}} x$ satisfies

$$V(x) \geq \epsilon \|E_{\mathrm{c}} x\|^2, \qquad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.58}$$
$$V(x) \geq \epsilon \|E_{\mathrm{c}} x\|^2, \qquad (t, x(t,\tau)) \in \mathcal{Z}, \tag{4.59}$$

for sufficiently small $\epsilon > 0$. Also, let $p > 1$. Whenever

$$X(x(t+\theta)) < pV(x(t)) \text{ for all } -\tau \leq \theta \leq 0,$$
$$(t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.60}$$
$$X(x(k+\theta)) < pV(x(k)) \text{ for all } -\tau \leq \theta \leq 0,$$
$$(t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.61}$$

is satisfied, we can calculate

$$\begin{aligned}
\dot{V}(x(t)) \quad &= \quad 2x^{\mathrm{T}}(t) E_{\mathrm{c}}^{\mathrm{T}} P [A_{\mathrm{c}} x(t) + \int_{-\tau}^{0} A_{\mathrm{c}1}(\theta) x(t+\theta)\mathrm{d}\theta] \\
&\leq 2x^{\mathrm{T}}(t) E_{\mathrm{c}}^{\mathrm{T}} P [A_{\mathrm{c}} x(t) + \int_{-\tau}^{0} A_{\mathrm{c}1}(\theta) x(t+\theta)\mathrm{d}\theta] \\
&\quad + \int_{-\tau}^{0} \alpha [p x^{\mathrm{T}}(t) E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} x(t) - x^{\mathrm{T}}(t+\theta) E_{\mathrm{c}}^{\mathrm{T}} p E_{\mathrm{c}} x(t+\theta)]\mathrm{d}\theta \\
&= x^{\mathrm{T}}(t)[A_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}} P A_{\mathrm{c}} + \int_{-\tau}^{0} R(\theta)\mathrm{d}\theta] x(t) \\
&\quad + \int_{\tau}^{0} \begin{pmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t+\theta) \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} p\alpha E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} - R(\theta) & E_{\mathrm{c}}^{\mathrm{T}} P A_{\mathrm{c}1}(\theta) \\ A_{\mathrm{c}1}(\theta)^{\mathrm{T}} P E_{\mathrm{c}} & -\alpha E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} \end{pmatrix} \begin{pmatrix} x(t) \\ x(t+\theta) \end{pmatrix} \mathrm{d}\theta, \qquad (t, x(t,\tau)) \notin \mathcal{Z},
\end{aligned} \tag{4.62}$$

$$\Delta V(x(k)) = V(x(k+1) - V(x(k)) =$$
$$x(k)^{\mathrm{T}} A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}} x(k) + x(k)^{\mathrm{T}} A_{\mathrm{d}}^{\mathrm{T}} P \sum A_{\mathrm{d}1} x(k+\theta)$$
$$+ \sum x(k+\theta)^{\mathrm{T}} A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}} x(k)$$
$$+ \sum x(k+\theta)^{\mathrm{T}} A_{\mathrm{d}}^{\mathrm{T}} P \sum A_{\mathrm{d}1} x(k+\theta) - x(k)^{\mathrm{T}} E_{\mathrm{d}} P E_{\mathrm{d}} x(k)$$
$$\leq x(k)^{\mathrm{T}} A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}} x(k) + x(k)^{\mathrm{T}} A_{\mathrm{d}}^{\mathrm{T}} P \sum A_{\mathrm{d}1} x(k+\theta)$$
$$+ \sum x(k+\theta)^{\mathrm{T}} A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}} x(k) + \sum x(k+\theta)^{\mathrm{T}} A_{\mathrm{d}}^{\mathrm{T}} P \sum A_{\mathrm{d}1} x(k+\theta)$$
$$-x(k)^{\mathrm{T}} E_{\mathrm{d}} P E_{\mathrm{d}} x(k) + \sum_{-\tau}^{0} \alpha [p x^{\mathrm{T}}(k) E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} x(t)$$
$$-x^{\mathrm{T}}(k+\theta) E_{\mathrm{d}}^{\mathrm{T}} p E_{\mathrm{d}} x(t+\theta)]$$
$$= x^{\mathrm{T}}(k) [A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}} - E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} + \sum_{-\tau}^{0} R_{\mathrm{d}}(\theta)] x(k)$$
$$+ \sum_{\tau}^{0} \begin{pmatrix} x^{\mathrm{T}}(k) & x^{\mathrm{T}}(k+\theta) \end{pmatrix}$$
$$\cdot \begin{pmatrix} p\alpha E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} - R_{\mathrm{d}}(\theta) & A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}1}(\theta) \\ A_{\mathrm{d}1}(\theta)^{\mathrm{T}} P A_{\mathrm{d}} & -\alpha E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} + A_{\mathrm{d}}^{\mathrm{T}} P \sum A_{\mathrm{d}1} \end{pmatrix} \begin{pmatrix} x(k) \\ x(k+\theta) \end{pmatrix} \mathrm{d}\theta, \qquad (t, x(t,\tau)) \in \mathcal{Z}, \tag{4.63}$$

With the above expression, and the fact that $p > 1$ can be arbitrarily close to 1, (4.54)–(4.55), and (4.56)–(4.57) imply

$$\dot{V}(x(t)) \leq -\epsilon \|E_{\mathrm{c}} x(t)\|^2, \qquad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{4.64}$$
$$\Delta V(x(k)) \leq -\epsilon \|E_{\mathrm{d}} x(t)\|^2, \qquad (t, x(t,\tau)) \in \mathcal{Z}, \tag{4.65}$$

for some sufficiently small $\epsilon > 0$. Therefore, the system is asymptotically stable according to Proposition 3.2. □

5 DELAY-DEPENDENT STABILITY CRITERIA BASED ON RAZUMIKHIN THEOREM

Consider again system (3.17)–(3.18) reformulated for this section for simplicity of exposition as:

$$E_{\mathrm{c}} \dot{x}(t) = A_{\mathrm{c}} x(t) + A_{\mathrm{c}1} x(t-\tau), \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{5.66}$$
$$E_{\mathrm{d}} \Delta x(k) = A_{\mathrm{d}} x(k) + A_{\mathrm{d}1} x(k-\tau), \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{5.67}$$

with initial condition (3.19), where the initial function is $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. With the observation that

$$E_{\mathrm{c}} x(t-\tau) = E_{\mathrm{c}} x(t) - \int_{-\tau}^{0} \dot{E}_{\mathrm{c}} x(t+\theta) \mathrm{d}\theta$$
$$= E_{\mathrm{c}} x(t) - \int_{-\tau}^{0} [A_{\mathrm{c}} x(t+\theta) + A_{\mathrm{c}1} x(t-\tau+\theta)] \mathrm{d}\theta,$$
$$(t, x(t,\tau)) \notin \mathcal{Z}, \tag{5.68}$$

and

$$E_{\mathrm{d}} x(k-\tau) = E_{\mathrm{d}} x(k) - \sum_{-\tau}^{0} \delta E_{\mathrm{d}} x(k+\theta)$$
$$= E_{\mathrm{d}} x(k) - \sum_{-\tau}^{0} [A_{\mathrm{d}} x(k+\theta) + A_{\mathrm{d}1} x(k-\tau+\theta)],$$
$$(t, x(t,\tau)) \in \mathcal{Z}, \tag{5.69}$$

for $t, k \geq \tau$, we can write the system (5.66)–(5.67) as

$$E_{\mathrm{c}} \dot{x}(t) = [A_{\mathrm{c}} + A_{\mathrm{c}1}] x(t)$$
$$+ \int_{-\tau}^{0} [-A_{\mathrm{c}1} A_{\mathrm{c}} x(t+\theta) = A_{\mathrm{c}1} A_{\mathrm{c}1}$$
$$x(t-\tau+\theta)] \mathrm{d}\theta, \qquad (t, x(t,\tau)) \in \mathcal{Z}, \tag{5.70}$$

and

$$\begin{aligned} &E_{\mathrm{d}}\Delta x(k) = [A_{\mathrm{d}} + A_{\mathrm{d1}}]x(k) \\ &+\sum_{-\tau}^{0}[-A_{\mathrm{d1}}A_{\mathrm{d}}x(k+\theta) = A_{\mathrm{d1}}A_{\mathrm{d1}} \\ &x(k-\tau+\theta)], \qquad (t, x(t,\tau)) \in \mathcal{Z}, \end{aligned} \tag{5.71}$$

with initial condition

$$x(\theta) = \psi(\theta), \quad = tau \leq \theta\tau \tag{5.72}$$

where

$$\psi(\theta) = \left\{ \begin{array}{cc} \phi(\theta) & -\tau \leq \theta \leq 0 \\ \text{solution of } (5.66) \text{ with i.c. } (3.17) & 0 < \theta \leq \tau \end{array}, \right\} \qquad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{5.73}$$

and

$$\psi(\theta) = \left\{ \begin{array}{cc} \phi(\theta) & -\tau \leq \theta \leq 0 \\ \text{solution of } (5.67) \text{ with i.c. } (3.18) & 0 < \theta \leq \tau \end{array} \right\} \tag{5.74}$$

Therefore, the system described by (3.17)–(3.18) and (3.19) is embedded in the system described by (5.70)–(5.71) and (5.72) without the initial condition constraint. Since this is a time-invariant system we can shift the initial time and write the system in a more standard form

$$\dot{y}(t) = \bar{A}_{\mathrm{c}}y(t) + \int_{-2\tau}^{0} \bar{A}_{\mathrm{c}}(\theta)y(t+\theta)\mathrm{d}\theta, \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{5.75}$$

$$\Delta y(k) = \bar{A}_{\mathrm{d}}y(k) + \sum_{-2\tau}^{0} \bar{A}_{\mathrm{d}}(\theta)y(k+\theta), \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{5.76}$$

(5.77)

where

$$\left\{ \begin{array}{ll} \bar{A}_{\mathrm{c}} = A_{\mathrm{c}} + A_{\mathrm{c1}} & \\ \bar{A}(\theta) = -A_{\mathrm{c1}}A_{\mathrm{c}} & \theta \in [-\tau, 0] \\ \bar{A}(\theta) = -A_{\mathrm{c1}}A_{\mathrm{c1}} & \theta \in [-2\tau, -\tau) \end{array} \right. \qquad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{5.78}$$

and

$$\left\{ \begin{array}{ll} \bar{A}_{\mathrm{d}} = A_{\mathrm{d}} + A_{\mathrm{d1}} & \\ \bar{A}_{\mathrm{d}}(\theta) = -A_{\mathrm{d1}}A_{\mathrm{d}} & \theta \in [-\tau, 0] \\ \bar{A}_{\mathrm{d}}(\theta) = -A_{\mathrm{d1}}A_{\mathrm{d1}} & \theta \in [-2\tau, -\tau) \end{array} \right. \qquad (t, x(t,\tau)) \in \mathcal{Z}, \tag{5.79}$$

with initial condition

$$y(\theta) = \psi(\theta), \quad -2\tau \leq \theta \leq 0. \tag{5.80}$$

The stability of the system represented by (5.75) to (5.80) implies the stability of the original system. This model transformation also introduces additional dynamics which are more complicated and will not be discussed here.

Since the transformed system is one with distributed delay, we can use Proposition 4.2 to derive the stability condition, which, of course, is sufficient for the stability of the original system.

Proposition 5.1. *The system described by (3.17)–(3.18) is asymptotically stable if there exist real scalars* $\alpha_0 > 0$ *,*

$\alpha_1 > 0$ *and real symmetric matrices* $P > 0$, R_0, R_1, *such that*

$$[E_{\rm c}^{\rm T}P(A_{\rm c}+A_{\rm c1})+(A_{\rm c}+A_{\rm c1})^{\rm T}PE_{\rm c}]+\tau(R_{\rm c0}+R_{\rm c1}) < 0, \quad (t,x(t,\tau)) \notin \mathcal{Z} \tag{5.81}$$

$$[A_{\rm d}^{\rm T}P(A_{\rm d}+A_{\rm d1})+(A_{\rm d}+A_{\rm d1})^{\rm T}PA_{\rm d}]+\tau(R_{\rm d0}+R_{\rm d1)<0,} \quad (t,x(t,\tau)) \in \mathcal{Z} \tag{5.82}$$

$$\begin{bmatrix} \alpha_k E_{\rm c}^{\rm T}PE_{\rm c} - R_k & -E_{\rm c}^{\rm T}PA_{\rm c1}A_k \\ A_k^{\rm T}A_{\rm c1}^{\rm T}PE_{\rm c} & -\alpha_k E_{\rm c}^{\rm T}PE_{\rm c} \end{bmatrix} < 0, k=0,1. \quad (t,x(t,\tau)) \notin \mathcal{Z}, \tag{5.83}$$

$$\begin{bmatrix} \alpha_k E_{\rm d}^{\rm T}PE_{\rm d} - R_{\rm dk} & -A_{\rm d}^{\rm T}PA_{\rm d1}A_{\rm dk} \\ A_{\rm dk}^{\rm T}A_{\rm d1}^{\rm T}PA_{\rm d} & -\alpha_k E_{\rm d}^{\rm T}PE_{\rm d} + A_{\rm d}^{\rm T}P\sum A_{\rm d1}, \end{bmatrix} < 0, k=0,1. \quad (t,x(t,\tau)) \in \mathcal{Z}, \tag{5.84}$$

Proof. We only need to prove the stability of the transformed system described by (5.75) to (5.79). Using Proposition 4.2, it can be concluded that the system is asymptotically stable if there exist $\alpha(\theta)$, P, and $R_{\rm c}(\theta)$, $R_{\rm d}(\theta)$ to satisfy

$$E_{\rm c}^{\rm T}P\bar{A}_{\rm c} + \bar{A}_{\rm c}{}^{\rm T}PE_{\rm c} + \int_{-2\tau}^{0} R_{\rm c}(\theta){\rm d}\theta < 0, \quad (t,x(t,\tau)) \notin \mathcal{Z}, \tag{5.85}$$

$$A_{\rm d}^{\rm T}P\bar{A}_{\rm d} + \bar{E}_{\rm d}{}^{\rm T}PE_{\rm d} + \sum_{-2\tau}^{0} R_{\rm d}(\theta) < 0, \quad (t,x(t,\tau)) \in \mathcal{Z}, \tag{5.86}$$

$$\begin{pmatrix} \alpha(\theta)E_{\rm c}^{\rm T}PE_{\rm c} - R_{\rm c}(\theta) & E_{\rm c}^{\rm T}PA(\theta) \\ A(\theta)^{\rm T}PE_{\rm c} & \alpha(\theta)E_{\rm c}^{\rm T}PE_{\rm c} \end{pmatrix} < 0, \quad -2\tau \le \theta < 0. \quad (t,x(t,\tau)) \notin \mathcal{Z}, \tag{5.87}$$

$$\begin{pmatrix} \alpha(\theta)E_{\rm d}^{\rm T}PE_{\rm d} - R_{\rm d}(\theta) & A_{\rm d}^{\rm T}PA(\theta) \\ A(\theta)^{\rm T}PA_{\rm d} & \alpha(\theta)E_{\rm d}^{\rm T}PE_{\rm d} + A_{\rm d}^{\rm T}P\sum A_{\rm d1} \end{pmatrix} < 0, \quad -2\tau \le \theta < 0. \quad (t,x(t,\tau)) \in \mathcal{Z}, \tag{5.88}$$

Choosing the following piecewise constant (matrix) functions

$$\alpha(\theta) = \begin{cases} \alpha_0 & -\tau \le \theta < 0, \\ \alpha_1 & -2\tau \le \theta < -\tau, \end{cases}, \quad (t,x(t,\tau)) \notin \mathcal{Z}, \tag{5.89}$$

$$R_{\rm c}(\theta) = \begin{cases} R_{\rm c0} & -\tau \le \theta < 0, \\ R_{\rm c1} & -2\tau \le \theta < -\tau, \end{cases}, \quad (t,x(t,\tau)) \notin \mathcal{Z} \tag{5.90}$$

and

$$\alpha(\theta) = \begin{cases} \alpha_0 & -\tau \le \theta < 0, \\ \alpha_1 & -2\tau \le \theta < -\tau, \end{cases}, \quad (t,x(t,\tau)) \in \mathcal{Z}, \tag{5.91}$$

$$R_{\rm d}(\theta) = \begin{cases} R_{\rm c0} & -\tau \le \theta < 0, \\ R_{\rm c1} & -2\tau \le \theta < -\tau, \end{cases}, \quad (t,x(t,\tau)) \in \mathcal{Z}, \tag{5.92}$$

completes the proof. □

The stability criterion in Proposition 5.1 depends on the time delay τ and is therefore delay-dependent. We can eliminate the arbitrary matrices R_0 and R_1 among the three matrix inequalities (5.81) to (5.84) to arrive at the following equivalent form.

Corollary 5.1. *The system described by (3.17)–(3.18) is asymptotically stable if there exist real symmetric matrix $P > 0$ and real scalars $alpha_0, \alpha_1 > 0$ such that*

$$\begin{pmatrix} M_{\rm c} & -E_{\rm c}^{\rm T} P A_{\rm c1} A_{\rm c} & -E_{\rm c}^{\rm T} P A_{\rm c1}^2 \\ -A_{\rm c}^{\rm T} A_{\rm c1}^{\rm T} P E_{\rm c} & -\alpha_0 E_{\rm c}^{\rm T} P E_{\rm c} & 0 \\ -(A_{\rm c1}^2)^{\rm T} P E_{\rm c} & 0 & -\alpha_1 E_{\rm c}^{\rm T} P E_{\rm c} \end{pmatrix} < 0, \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{5.93}$$

and

$$\begin{pmatrix} M_{\rm d} & -A_{\rm d}^{\rm T} P A_{\rm d1} A_{\rm d} & -A_{\rm d}^{\rm T} P A_{\rm d1}^2 \\ -A_{\rm d}^{\rm T} A_{\rm d1}^{\rm T} P A_{\rm d} & -\alpha_0 E_{\rm d}^{\rm T} P E_{\rm d} & 0 \\ -(A_{\rm d1}^2)^{\rm T} P E_{\rm d} & 0 & -\alpha_1 E_{\rm d}^{\rm T} P E_{\rm d} \end{pmatrix} < 0, \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{5.94}$$

where

$$M_{\rm c} = \frac{1}{\tau}[E_{\rm c}^{\rm T} P (A_{\rm c} + A_{\rm c1}) + (A_{\rm c} + A_{\rm c1})^{\rm T} P E_{\rm c}] + (\alpha_0 + \alpha_1) P, \quad (t, x(t,\tau)) \notin \mathcal{Z}, \tag{5.95}$$

$$M_{\rm d} = \frac{1}{\tau}[A_{\rm d}^{\rm T} P (A_{\rm d} + A_{\rm d1}) + (A_{\rm d} + A_{\rm d1})^{\rm T} P A_{\rm d}] + (\alpha_0 + \alpha_1) P, \quad (t, x(t,\tau)) \in \mathcal{Z}, \tag{5.96}$$

Proof. Start with dividing (5.81)–(5.82) by τ, and eliminating $R_{\rm c0}$ and $R_{\rm d0}$ among the resulting matrix inequality and (5.83)–(5.84) for $k = 1$. Details of proof are left for exercise. □

6 CONCLUSION

In this paper we have presented new class of singularly impulsive dynamical systems with time delay. For this class systems we have developed asymptotic stability results by using Razumikhin stability theorem.

7 FUTURE WORK

In future work we will develop stability results for the class of singularly impulsive dynamical systems based on Lyapunov - Krasovskii stability theorem.

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