The 6th International Federation of Nonlinear Analysts Conference (2012) 57–66

Practical Stability of Singularly Impulsive Dynamical Systems: Bellman-Gronwall Approach

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ABSTRACT: In this paper we present *new* model of *singularly impulsive dynamical systems*. Dynamics of this system is characterized by the set of differential, difference, and algebraic equations. They represent the class of hybrid systems, where algebraic equations represent constraints that differential and difference equations need to satisfy. For the class of singularly impulsive dynamical systems we state and prove Bellman-Gronwall lemma. Furthermore, using Bellman-Gronwall lemma for the class of singularly impulsive dynamical systems we present stability results.

1 INTRODUCTION

Modern complex engineering systems as well as biological and physiological systems typically possess a multi-echelon hierarchical hybrid architecture characterized by continuous-time dynamics at the lower levels of hierarchy and discrete-time dynamics at the higher levels of the hierarchy. Hence, it is not surprising that hybrid systems have been the subject of intensive research over the past recent years (see Branicky et al. 1998, Ye et al. 1998 b, Haddad, Chellaboina and Kablar 2001a-b). Such systems include dynamical switching systems (Branicky 1998, Leonessa et al. 2000), nonsmooth impact and constrained mechanical systems (Back et al. 1993, Brogliato 1996, Brogliato et al. 1997), biological systems (Lakshmikantham et al. 1989), demographic systems (Liu 1994), sampled-data systems (Hagiwara and Araki 1988), discrete-event systems (Passino et al. 1994), intelligent vehicle/highway systems (Lygeros et al. 1998) and flight control systems (Tomlin et al. 1998), etc. The mathematical descriptions of many of these systems can be characterized by impulsive differential equations (Simeonov and Bainov 1985, 1987, Liu 1988, Lakshmikantham et al. 1989, 1994, Bainov and Simeonov 1989, 1995, Kulev and Bainov 1989, Lakshmikantham and Liu 1989, Hu et al. 1989, Samoilenko and Perestyuk 1995, Haddad, Chellaboina and Kablar 2001a-b). Impulsive dynamical systems can be viewed as a subclass of hybrid systems.

Motivated by the results on impulsive dynamical systems presented in Haddad, Chellaboina, and Kablar (2001a-b) and Haddad, Kablar, and Chellaboina (2000, 2005) and the authors previous work on singular or generalized systems, we present new class of *singularly impulsive* or *generalized impulsive dynamical systems*. It presents novel class of hybrid systems and generalization of impulsive dynamical systems to incorporate singular nature of the systems. Extensive applications of this class of systems can be found in contact problems and in hybrid systems.

We present mathematical model of the singularly impulsive dynamical systems. We show how it can be viewed as general systems from which impulsive dynamical systems, singular continuous-time and singular dicrete-time systems follow. Then we present Assumptions needed for the model and the division of this class of systems to time-dependent and state-dependent singularly impulsive dynamical systems with respect to the resetting set. We present examples of mathematical and physical models, and then we draw some conclusions and define future work.

At first, we establish definitions and notations. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, let $\mathcal{N}$ denote the set of nonnegative integers, and let I_n or I denote the $n \times n$ identity matrix. Furthermore, let $\partial\mathcal{S}, \dot{\mathcal{S}}, \bar{\mathcal{S}}$ denote the boundary, the interior, and a closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. Finally, let C^0 denote the set of continuous functions and C^r denote the set of functions with r continuous derivatives.

2 MATHEMATICAL MODEL OF SINGULARLY IMPULSIVE DYNAMICAL SYSTEMS

A singularly impulsive dynamical system consists of three elements:

1. A possibly singular continuous-time dynamical equation, which governs the motion of the system between resetting events;
2. A possibly singular difference equation, which governs the way the states are instantaneously changed when a resetting occurs; and
3. A criterion for determining when the states of the system are to be reset.

Received November 30, 2012 IFNA Publishers, Inc.

Mathematical model of these systems is described with

$$E_{\rm c}\dot{x}(t) = f_{\rm c}(x(t))+G_{\rm c}(x(t))u_{\rm c}(t), \quad (t,x(t),u_{\rm c}(t))\notin\mathcal{S}, \tag{1}$$

$$E_{\rm d}\triangle x(t) = f_{\rm d}(x(t))+G_{\rm d}(x(t))u_{\rm d}(t), \quad (t,x(t),u_{\rm c}(t))\in\mathcal{S}, \tag{2}$$

$$y_{\rm c}(t) = h_{\rm c}(x(t))+J_{\rm c}(x(t))u_{\rm c}(t), \quad (t,x(t),u_{\rm c}(t))\notin\mathcal{S}, \tag{3}$$

$$y_{d}(t) = h_{\rm d}(x(t))+J_{\rm d}(x(t))u_{\rm d}(t), \quad (t,x(t),u_{\rm c}(t))\in\mathcal{S}, \tag{4}$$

where $t \geq 0$, $x(0) = x_0$, $x(t) \in \mathcal{D} \subset \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u_{\rm c} \in \mathcal{U}_{\rm c} \subset \mathbb{R}^{m_{\rm c}}$,$u_{\rm d}(t_k) \in \mathcal{U}_{\rm d} \subset \mathbb{R}^{m_{\rm d}}$, t_k denotes $k^{\rm th}$ instant of time at which $(t, x(t), u_{\rm c}(t))$ intersects $\mathcal{S}$ for a particular trajectory $x(t)$ and input $u_{\rm c}(t), y_{\rm c}(t) \in \mathbb{R}^{l_{\rm c}}, y_{\rm d}(t_k) \in \mathbb{R}^{l_{\rm d}}$, $f_{\rm c} : \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $f_{\rm c}(0) = 0$, $G_{\rm c} : \mathcal{D} \to \ltimes \times \gtrdot_{\rm c}$, $f_{\rm d} : \mathcal{D} \to \mathbb{R}^n$ is continuous and satisfies $f_{\rm d}(0) = 0$, $G_{\rm d} : \mathcal{D} \to \mathbb{R}^{n\times m_{\rm d}}$, $h_{\rm c} : \mathcal{D} \to \mathbb{R}^{l_{\rm c}}$ and satisfies $h_{\rm c}(0) = 0$, $J_{\rm c} : \mathcal{D} \to \mathbb{R}^{l_{\rm c}\times m_{\rm c}}$, $h_{\rm d} : \mathcal{D} \to \mathbb{R}^{l_{\rm d}}$ and satisfies $h_{\rm d}(0) = 0$, $J_{\rm d} : \mathcal{D} \to \mathbb{R}^{l_{\rm d}\times m_{\rm d}}$, and $\mathcal{S} \subset [0,\infty) \times \mathbb{R}^n \times \mathcal{U}_{\rm c}$ is the *resetting set*. Here, as in Haddad, Chellaboina, and Kablar (2001a) we assume that $u_{\rm c}(\cdot)$ and $u_{\rm d}(\cdot)$ are restricted to the class of *admissible* inputs consisting of measurable functions $(u_{\rm c}(t), u_{\rm d}(t)) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$ for all $t \geq 0$ and $k \in \mathcal{N}_{[0,t)} \equiv k : 0 \leq t_k < t$, where the constraint set $\mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$ is given with $(0, 0) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$. We refer to the differential equation (1) as the *continuous-time dynamics*, and we refer to the difference equation (2) as the *resetting law*.

Matrices $E_{\rm c}$, $E_{\rm d}$ may be singular matrices. In case $E_{\rm c} = I$, $E_{\rm d} = I$, (1)–(4) represent standard impulsive dynamical systems described in Haddad, Chellaboina, and Kablar (2001a), and Haddad, Kablar, and Chellaboina (2000, 2005), where stability, dissipativity, feedback interconnections, optimality, robustness, and disturbance rejection has been analyzed. In absence of discrete dynamics they specialize to singular continuous-time systems, with further specialization $E_{\rm c} = I$ to standard continuous-time systems. If only discrete dynamics is present they specialize to singular discrete-time systems, with further specialization $E_{\rm d} = I$ to standard discrete-time systems. Therefore, theory of the singularly impulsive or generalized impulsive dynamical systems once developed, can be viewed as a generalization of the singular and impulsive dynamical system theory, unifying them into more general new system theory.

In what follows is given basic setting and division of this class of systems with respect to the definition of the resetting sets, accompanied with adequate assumptions needed for the model.

We make the following additional assumptions:

A1. $(0, x_0, u_{\rm c0}) \notin \mathcal{S}$, where $x(0) = x_0$ and $u_{\rm c}(0) = u_{\rm c0}$, that is, the initial condition is not in $\mathcal{S}$.

A2. If $(t, x(t), u_{\rm c}(t)) \in \bar{\mathcal{S}}\backslash\mathcal{S}$ then there exists $\epsilon > 0$ such that, for all $0 < \delta < \epsilon$, $s(t + \delta; t, x(t), u_{\rm c}(t + \delta)) \notin \mathcal{S}$.

A3. If $(t_k, x(t_k), u_{\rm c}(t_k)) \in \partial S \cap \mathcal{S}$ then there exists $\epsilon > 0$ such that, for all $0 < \delta < \epsilon$ and $u_{\rm d}(t_k) \in \mathcal{U}_{\rm d}$, $s(t_k + \delta; t_k, E_{\rm d}x(t_k) + f_{\rm d}(x(t_k)) + G_{\rm d}(x(t_k))u_{\rm d}(t_k), u_{\rm c}(t_k + \delta)) \notin \mathcal{S}$.

A4. We assume consistent initial conditions (and prior and after every resetting).

Assumption A1 ensures that the initial condition for the resetting differential equation (1), (2) is not a point of discontinuity, and this assumption is made for convenience. If $(0, x_0, u_{\rm c0}) \in \mathcal{S}$, then the system initially resets to $E_{\rm d}x_0^+ = E_{\rm d}x_0 + f_{\rm d}(x_0) + G_{\rm d}(x_0)u_{\rm d}(0)$ which serves as the initial condition for the continuous dynamics (1). It follows from A3 that the trajectory then leaves $\mathcal{S}$. We assume in A2 that if a trajectory reaches the closure of $\mathcal{S}$ at a point that does not belong to $\mathcal{S}$, then the trajectory must be directed away from $\mathcal{S}$, that is, a trajectory cannot enter $\mathcal{S}$ through a point that belongs to the closure of $\mathcal{S}$ but not to $\mathcal{S}$. Finally, A3 ensures that when a trajectory intersects the resetting set $\mathcal{S}$, it instantaneously exits $\mathcal{S}$, see Figure 1. We make the following remarks.

Figure 1. Resetting Set.

Remark 2.1. *It follows from A3 that resetting removes the pair $(t_k, x_k, u_{\rm c}(t_k))$ from the resetting set $\mathcal{S}$. Thus, immediately after resetting occurs, the continuous-time dynamics (1), and not the resetting law (2), becomes the active element of the singularly impulsive dynamical system, Haddad, Chellaboina and Kablar (2001a), Kablar (2003).*

Remark 2.2. *It follows from A1-A3 that no trajectory can intersect the interior of $\mathcal{S}$. According to $A1$, the trajectory $x(t)$ begins outside the set $\mathcal{S}$. Furthermore, it follows from A2 that a trajectory can only reach $\mathcal{S}$ through a point belonging to both $\mathcal{S}$ and its boundary. Finally, from A3, it follows that if a trajectory reaches a point $\mathcal{S}$ that is on the boundary of $\mathcal{S}$, then the trajectory is instantaneously removed from $\mathcal{S}$. Since a continuous trajectory starting outside of $\mathcal{S}$ and intersecting the interior of $\mathcal{S}$ must first intersect the boundary of $\mathcal{S}$, it follows that no trajectory can reach the interior of $\mathcal{S}$, Haddad, Chellaboina and Kablar (2001a), Kablar (2003).*

Remark 2.3. *It follows from A1-A3 and Remark 1.2 that $\partial\mathcal{S} \cup \mathcal{S}$ is closed and hence the resetting times t_k are well defined and distinct, Haddad, Chellaboina and Kablar (2001a), Kablar (2003).*

Remark 2.4. *Since the resetting times are well defined and distinct, and since the solutions to (1) exist and are unique, it follows that the solutions of the singularly impulsive dynamical system (1), (2) also exist and are unique over a forward time interval, Haddad, Chellaboina and Kablar (2001a), Kablar (2003).*

In Haddad, Chellaboina and Kablar (2001a), the resetting set $\mathcal{S}$ is defined in terms of a countable number of functions $\tau_k : \mathbb{R}^n \to (0,\infty)$, and is given by

$$\mathcal{S} = \cup_k\{(\tau_k(x), x, u_{\rm c}(\tau_k(x)) : x \in \mathbb{R}^n\}. \tag{5}$$

The analysis of singularly impulsive dynamical systems with a resetting set of the form (5) can be quite involved. In particular, such systems exhibit Zenoness, beating, as well as confluence phenomena wherein solutions exhibit infinitely many transitions in a finite times, and coincide after a given point of time, Haddad, Chellaboina and Kablar (2001a). In this chapter we assume that existence and uniqueness properties of a given singularly impulsive dynamical system are satisfied in forward time. Furthermore, since singularly impulsive dynamical systems of the form (1)-(4) involve impulses at variable times they are time-varying systems.

Here we will consider singularly impulsive dynamical systems involving two distinct forms of the resetting set $\mathcal{S}$. In the first case, the resetting set is defined by a prescribed sequence of times which are independent of state x. These equations are thus called *time-dependent singularly impulsive dynamical systems*. In the second case, the resetting set is defined by a region in the state space that is independent of time. These equations are called *state-dependent singularly impulsive dynamical systems*.

2.1 Time-Dependent Singularly Impulsive Dynamical Systems

Time-dependent singularly impulsive dynamical systems can be written as (1)–(4) with $\mathcal{S}$ defined as

$$\mathcal{S} = \tau \times \mathbb{R}^n \times \mathcal{U}_\mathrm{c}, \tag{6}$$

where

$$\tau = t_1, t_2, \ldots \tag{7}$$

and $0 < t_1 < t_2 < \ldots$ are prescribed resetting times. When an infinite number of resetting times are used and $t_k \to \infty$ as $k \to \infty$, then $\mathcal{S}$ is closed. Now (1)–(4) can be rewritten in the form of the *time-dependent singularly impulsive dynamical system*

$$E_\mathrm{c}\dot{x}(t) = f_\mathrm{c}(x(t)) + G_\mathrm{c}(x(t))u_\mathrm{c}(t), \quad t \neq t_k, \tag{8}$$

$$E_\mathrm{d}\triangle x(t) = f_\mathrm{d}(x(t)) + G_\mathrm{d}(x(t))u_\mathrm{d}(t), \quad t = t_k, \tag{9}$$

$$y_\mathrm{c}(t) = h_\mathrm{c}(x(t)) + J_\mathrm{c}(x(t))u_\mathrm{c}(t), \quad t \neq t_k, \tag{10}$$

$$y_d(t) = h_\mathrm{d}(x(t)) + J_\mathrm{d}(x(t))u_\mathrm{d}(t), \quad t = t_k. \tag{11}$$

Since $0 \notin \tau$ and $t_k < t_{k+1}$, it follows that the assumptions A1–A3 are satisfied. Since time-dependent singularly impulsive dynamical systems involve impulses at a fixed sequence of times, they are time-varying systems.

Remark 2.5. *Standard continuous-time and discrete-time dynamical systems as well as sampled-data systems can be treated as special cases of singularly impulsive dynamical systems. For details see [1].*

Remark 2.6. *The time-dependent singularly impulsive dynamical system (8)–(11), with $E_\mathrm{c} = I$ and $E_\mathrm{d} = I$ includes as a special case the impulsive control problem addressed in the literature wherein at least one of the state variables of the continuous-time plant can be changed instantaneously to any given value given by an impulsive control at a set of control instants τ, Haddad, Chellaboina and Kablar (2001a).*

2.2 State-Dependent Singularly Impulsive Dynamical Systems

State-dependent singularly impulsive dynamical systems can be written as (1)–(4) with $\mathcal{S}$ defined as

$$\mathcal{S} = [0, \infty) \times \mathcal{Z}, \tag{12}$$

where $\mathcal{Z} = \mathcal{Z}_x \times \mathcal{U}_\mathrm{c}$ and $\mathcal{Z}_x \subset \mathbb{R}^n$. Therefore, (1)–(4) can be rewritten in the form of the *state-dependent singularly impulsive dynamical system*

$$E_\mathrm{c}\dot{x}(t) = f_\mathrm{c}(x(t)) + G_\mathrm{c}(x(t))u_\mathrm{c}(t), \quad (x(t), u_\mathrm{c}(t)) \notin \mathcal{Z}, \tag{13}$$

$$E_\mathrm{d}\triangle x(t) = f_\mathrm{d}(x(t)) + G_\mathrm{d}(x(t))u_\mathrm{d}(t), \quad (x(t), u_\mathrm{c}(t)) \in \mathcal{Z}, \tag{14}$$

$$y_\mathrm{c}(t) = h_\mathrm{c}(x(t)) + J_\mathrm{c}(x(t))u_\mathrm{c}(t), \quad (x(t), u_\mathrm{c}(t)) \notin \mathcal{Z}, \tag{15}$$

$$y_d(t) = h_\mathrm{d}(x(t)) + J_\mathrm{d}(x(t))u_\mathrm{d}(t), \quad (x(t), u_\mathrm{c}(t)) \in \mathcal{Z}. \tag{16}$$

We assume that $(x_0, u_{\mathrm{c}0}) \notin \mathcal{Z}$, $(0, 0) \notin \mathcal{Z}$, and that the resetting action removes the pair (x, u_c) from the set $\mathcal{Z}$; that is, if $(x, u_\mathrm{c}) \in \mathcal{Z}$ then $(E_\mathrm{d}x + f_\mathrm{d}(x) + G_\mathrm{d}(x)u_\mathrm{d}, u_\mathrm{c}) \notin \mathcal{Z}$, $u_\mathrm{d} \in \mathcal{U}_\mathrm{d}$. In addition, we assume that if at time t the trajectory $(x(t), u_\mathrm{c}(t)) \in \bar{\mathcal{Z}} \backslash \mathcal{Z}$, then there exists $\epsilon > 0$ such that for $0 < \delta < \epsilon$, $(x(t + \delta), u_\mathrm{c}(t + \delta)) \notin \mathcal{Z}$.

These assumptions represent the specialization of A1–A3 for the particular resetting set (12). It follows from these assumptions that for a particular initial condition, the resetting times $\tau_k(x_0)$ are distinct and well defined. Since the resetting set $\mathcal{Z}$ is a subset of the state space and is independent of time, state-dependent singularly impulsive dynamical systems are time-invariant systems. Finally, in the case where $\mathcal{S} \equiv [0, \infty) \times \mathbb{R}^n \times \mathcal{Z}_{u_\mathrm{c}}$, where $\mathcal{Z}_{u_\mathrm{c}} \subset \mathcal{U}_\mathrm{c}$ we refer to (13)–(16) as an input-dependent singularly impulsive dynamical system. Both these cases represent a generalization to the impulsive control problem considered in the literature.

3 BELLMAN - GRONWALL LEMMA FOR SINGULARLY IMPULSIVE DYNAMICAL SYSTEMS

Let us consider linear singularly impulsive dynamical system in free regime, given by

$$E_{\mathrm{c}}\dot{x}(t) = A_{\mathrm{c}}(x(t)), \quad (t, x(t)) \notin \mathcal{S}, \tag{17}$$

$$E_{\mathrm{d}}\triangle x(t) = A_{\mathrm{d}}(x(t)), \quad (t, x(t)) \in \mathcal{S}, \tag{18}$$

$$y_{\mathrm{c}}(t) = C_{\mathrm{c}}(x(t)), \quad (t, x(t)) \notin \mathcal{S}, \tag{19}$$

$$y_{d}(t) = C_{\mathrm{d}}(x(t)), \quad (t, x(t)) \in \mathcal{S}, \tag{20}$$

Solution of (17) is given by

$$x(t) = \Phi(t, t_0)\hat{E}\hat{E}^D x_0, \qquad x_0 \in \mathbb{N}(I - \hat{E}\hat{E}^D), \tag{21}$$

where

$$\Phi(t, t_0) = e^{\hat{E}^D \hat{A}(t-t_0)}, \tag{22}$$

is *fundamental matrix* of the system with corresponding matrices defined by,

$$\hat{E}_{\mathrm{c}} = (cE_{\mathrm{c}} - A_{\mathrm{c}})^{-1}E_{\mathrm{c}}, \;\; \hat{A}_{\mathrm{c}} = (cE_{\mathrm{c}} - A_{\mathrm{c}})^{-1}A_{\mathrm{c}}, \;\; c \in \mathbb{C}. \tag{23}$$

Solution of (18) is given by

$$x(t) = \Phi(k, k_0)\hat{E}\hat{E}^D x_0, \qquad x_0 \in \mathbb{N}(I - \hat{E}\hat{E}^D), \quad k \geq 1, \tag{24}$$

where

$$\Phi(t, t_0) = (\hat{E}^D \hat{A})^k, \tag{25}$$

is *fundamental matrix* of the system with corresponding matrices defined by,

$$\hat{E}_{\mathrm{c}} = (cE_{\mathrm{c}} - A_{\mathrm{c}})^{-1}E_{\mathrm{c}}, \;\; \hat{A}_{\mathrm{c}} = (cE_{\mathrm{c}} - A_{\mathrm{c}})^{-1}A_{\mathrm{c}}, \;\; c \in \mathbb{C}. \tag{26}$$

Lemma 3.1. *Suppose that there exist vector $q(t, t_0)$ defined by,*

$$q(t, t_0) = \Phi(t, t_0)\hat{E}\hat{E}^D v(t_0), q(k, h) = \Phi(k, h)\hat{E}\hat{E}^D v(h), \tag{27}$$

If,

$$\begin{gathered} E_{\mathrm{c}} q(t, t_0) = E_{\mathrm{c}}\Phi(t, t_0)\hat{E_{\mathrm{c}}}\hat{E_{\mathrm{c}}}^D v(t_0), \\ (t, x(t)) \notin \mathcal{Z}, \end{gathered} \tag{28}$$

$$\begin{gathered} E_{\mathrm{d}} q(k, h) = E_{\mathrm{d}}\Phi(k, h)\hat{E_{\mathrm{d}}}\hat{E_{\mathrm{d}}}^D v(h), \\ (t, x(t)) \in \mathcal{Z}, \end{gathered} \tag{29}$$

then,

$$\begin{gathered} \|q(t, t_0)\|^2_{E_{\mathrm{c}}^T E_{\mathrm{c}}} \leq \|v(t_0)\|^2_{E_{\mathrm{c}}^T E_{\mathrm{c}}} e^{\Lambda_{\max}(M_{\mathrm{c}})(t-t_0)}, \\ (t, x(t)) \notin \mathcal{Z}, \end{gathered} \tag{30}$$

$$\begin{gathered} \|q(k, h)\|^2_{E_{\mathrm{d}}^T E_{\mathrm{d}}} \leq \|v(h)\|^2_{E_{\mathrm{d}}^T E_{\mathrm{d}}} \prod_{j=h}^{j=k-1} \Lambda_{\max}(A_{\mathrm{d}}^{\mathrm{T}} A_{\mathrm{d}}), \\ (t, x(t)) \in \mathcal{Z}, \end{gathered} \tag{31}$$

where,

$$\begin{gathered} \Lambda_{\max}(M_{\mathrm{c}}) = \max q^T(t, t_0) M_{\mathrm{c}} q(t, t_0) : \\ q(t, t_0) \in W_k \backslash 0, q(t, t_0) E_{\mathrm{c}}^T E_{\mathrm{c}} q(t, t_0) = 1, \\ (t, x(t)) \notin \mathcal{Z}, \end{gathered} \tag{32}$$

$$\begin{gathered} \Lambda_{\max}(A_{\mathrm{d}}^{\mathrm{T}} A_{\mathrm{d}}) = \max q^T(k, h) A_{\mathrm{d}}^{\mathrm{T}} A_{\mathrm{d}} q(k, h) : \\ q(k, h) \in W_k \backslash 0, q(k, h) E_{\mathrm{d}}^T E_{\mathrm{d}} q(k, h) = 1, \\ (t, x(t)) \in \mathcal{Z}, \end{gathered} \tag{33}$$

and,

$$M_{\mathrm{c}} = A_{\mathrm{c}}^{\mathrm{T}} E_{\mathrm{c}} + E_{\mathrm{c}}^T A_{\mathrm{c}}, \tag{34}$$

where,

$$v(t_0) = q(t, t_0), \qquad (t, x(t)) \notin \mathcal{Z}, \tag{35}$$

$$v(h) = q(k, h), \qquad (t, x(t)) \in \mathcal{Z}, \tag{36}$$

Proof. Differentiating by time equation 81,

$$\dot{x}(t) = (\frac{d}{dt}\Phi(t,t_0))\hat{E}_{\mathrm{c}}\hat{E}_{\mathrm{c}}^{\,D}x_0,\ (t,x(t)) \notin \mathcal{Z}, \tag{37}$$

$$q(k+1,h) = \Phi(k+1,h)\hat{E}_{\mathrm{d}}\hat{E}_{\mathrm{d}}^{\,D}v(h),\ (t,x(t)) \in \mathcal{Z}, \tag{38}$$

and multiplying the equation 37 with $E_{\mathrm{c,d}}$ from the left side,

$$\begin{gathered} E_{\mathrm{c}}\dot{x}(t) = E_{\mathrm{c}}(\tfrac{d}{dt}\Phi(t,t_0))\hat{E}_{\mathrm{c}}\hat{E}_{\mathrm{c}}^{\,D}x_0 = \\ A_{\mathrm{c}}x(t) = A_{\mathrm{c}}\Phi(t,t_0)\hat{E}_{\mathrm{c}}\hat{E}c^{\,D}x_0, \\ (t,x(t)) \notin \mathcal{Z}, \end{gathered} \tag{39}$$

$$\begin{gathered} E_{\mathrm{d}}q(k+1,h) = E_{\mathrm{d}}\Phi(k+1,h))\hat{E}_{\mathrm{d}}\hat{E}_{\mathrm{d}}^{\,D}v(h) = \\ A_{\mathrm{d}}x(k) = A_{\mathrm{d}}\Phi(k,h)\hat{E}_{\mathrm{d}}\hat{E}_{\mathrm{d}}^{\,D}x_0, \\ (t,x(t)) \in \mathcal{Z}, \end{gathered} \tag{40}$$

yields,

$$\begin{gathered} E_{\mathrm{c}}(\tfrac{d}{dt}\Phi(t,t_0))\hat{E}_{\mathrm{c}}\hat{E}_{\mathrm{c}}^{\,D} = A_{\mathrm{c}}\Phi(t,t_0)\hat{E}_{\mathrm{c}}\hat{E}_{\mathrm{c}}^{\,D}, \\ (t,x(t)) \notin \mathcal{Z}, \end{gathered} \tag{41}$$

$$\begin{gathered} E_{\mathrm{d}}\Phi(k+1,h))\hat{E}_{\mathrm{d}}\hat{E}_{\mathrm{d}}^{\,D} = A_{\mathrm{d}}\Phi(k,h)\hat{E}_{\mathrm{d}}\hat{E}_{\mathrm{d}}^{\,D}, \\ (t,x(t)) \in \mathcal{Z}, \end{gathered} \tag{42}$$

By differentiating over time equation 65,

$$\begin{gathered} E_{\mathrm{c}}\dot{q}(t,t_0) = E_{\mathrm{c}}(\tfrac{d}{dt}\Phi(t,t_0))\hat{E}\hat{E}^{D}v(t_0) = \\ A_{\mathrm{c}}x(t) = A_{\mathrm{c}}\Phi(t,t_0)\hat{E}_{\mathrm{d}}\hat{E}_{\mathrm{d}}^{\,D}x(t_0), \end{gathered} \tag{43}$$

$$\begin{gathered} E_{\mathrm{d}}q(k+1,h) = E_{\mathrm{d}}\Phi(k+1,h))\hat{E}_{\mathrm{d}}\hat{E}_{\mathrm{d}}^{\,D}v(h) = \\ A_{\mathrm{d}}x(t) = A_{\mathrm{d}}\Phi(k,h)\hat{E}_{\mathrm{d}}\hat{E}_{\mathrm{d}}^{\,D}x(h), \quad (t,x(t)) \in \mathcal{Z}, \end{gathered} \tag{44}$$

from where,

$$E_{\mathrm{c}}\dot{q}(t,t_0) = A_{\mathrm{c}}q(t,t_0).E_{\mathrm{d}}q(k+1,h) = A_{\mathrm{d}}q(k,h) \tag{45}$$

Forming quadratic form and differentiating over time, for $(t,x(t)) \notin \mathcal{Z}$,

$$\begin{aligned} &\frac{\partial}{\partial t}(q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}}q(t,t_0)) \\ &\quad = \frac{\partial q^T(t,t_0)}{\partial t}E_{\mathrm{c}}^T E_{\mathrm{c}}q(t,t_0)) + q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}}\frac{\partial q(t,t_0)}{\partial t} \\ &\quad = q(t,t_0)A_{\mathrm{c}}E_{\mathrm{c}}^T E_{\mathrm{c}}q(t,t_0)) + q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}}A_{\mathrm{c}}q(t,t_0) \\ &\quad = q(t,t_0)(A_{\mathrm{c}}E_{\mathrm{c}}^T + E_{\mathrm{c}}^T A_{\mathrm{c}})q(t,t_0) \\ &\quad \le \Lambda_{\max}(M_{\mathrm{c}})q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}}q(t,t_0) \\ &\quad \le \Lambda_{\max}(M_{\mathrm{c}})v(t,t_0) \end{aligned} \tag{46}$$

and forming quadratic form and looking for difference, for $(t,x(t)) \in \mathcal{Z}$,

$$\begin{aligned} &\Delta ln(q^T(k,h)E_{\mathrm{d}}^T E_{\mathrm{d}}q(k,h)) \\ &\quad = lnq^T(k+1,h)E_{\mathrm{d}}^T E_{\mathrm{d}}q(k+1,h) \\ &\quad -lnq^T(k,h)E_{\mathrm{d}}^T E_{\mathrm{d}}q(k,h) \\ &\quad = \frac{lnq^T(k+1,h)E_{\mathrm{d}}^T E_{\mathrm{d}}q(k+1,h)}{q^T(k,h)E_{\mathrm{d}}^T E_{\mathrm{d}}q(k,h)} \\ &\quad = \frac{lnq^T(k,h)A_{\mathrm{d}}^T A_{\mathrm{d}}q(k,h)}{q^T(k,h)E_{\mathrm{d}}^T E_{\mathrm{d}}q(k,h)} \\ &\quad \le \Lambda_{\max}(A_{\mathrm{d}}^{\mathrm{T}}A_{\mathrm{d}}) \end{aligned} \tag{47}$$

where,

$$v(t,t_0) = q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}}q(t,t_0), \tag{48}$$

$$v(k,h) = q^T(k,h)E_{\mathrm{d}}^T E_{\mathrm{d}}q(k,h), \tag{49}$$

where $\Lambda_{\max}(\cdot)$ and matrix M_{c} are defined with 70 and 71, and 33, respectively.

Finally, for $(t, x(t)) \notin \mathcal{Z}$,

$$\frac{\partial}{\partial t}(q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}} q(t,t_0)) = \\ \leq \Lambda_{\max}(M_{\mathrm{c}}) q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}} q(t,t_0). \tag{50}$$

Let us prepare, for $(t, x(t)) \in \mathcal{Z}$ holds,

$$q^{\mathrm{T}}(h,h) = q(h,h)v^{\mathrm{T}}(h)v(h) \tag{51}$$

and

$$\Phi(k,h) = \Phi(h,h) = I, \qquad q(h,h) = v(h). \tag{52}$$

Integrating 46 and summing 47,

$$\int_{t_0}^{t} \frac{d(q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}} q(t,t_0))}{q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}} q(t,t_0)} \\ \leq \int_{t_0}^{t} \Lambda_{\max}(M_{\mathrm{c}})dt, \qquad (t,x(t)) \notin \mathcal{Z}$$

$$\sum_{j=h}^{j=k-1} \Delta ln(q^{\mathrm{T}}(k,h)E_{\mathrm{d}}^{\mathrm{T}} E_{\mathrm{d}} q(k,h)) \\ \leq \sum_{j=h}^{j=k-1} ln\Lambda_{\max}(A_{\mathrm{d}}^{\mathrm{T}} A_{\mathrm{d}}), \qquad (t,x(t)) \in \mathcal{Z} \tag{53}$$

$$\tag{54}$$

and solving,

$$ln(q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}} q(t,t_0))|_{t_0}^{t} \leq \Lambda_{\max}(M_{\mathrm{c}})(t-t_0), \\ (t,x(t)) \notin \mathcal{Z}, \tag{55}$$

$$ln\frac{q^T(k,h)E_{\mathrm{d}}^T E_{\mathrm{d}} q(k,h)}{q^T(h)E_{\mathrm{d}}^T E_{\mathrm{d}} q(h)} \leq \\ \prod_{j=h}^{j=k-1} \Lambda_{\max}(A_{\mathrm{d}}^{\mathrm{T}} A_{\mathrm{d}}), \qquad (t,x(t)) \notin \mathcal{Z}, \tag{56}$$

i.e.,

$$q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}} q(t,t_0) \leq q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}} q(t,t_0) \\ e^{\Lambda_{\max}(M_{\mathrm{c}})(t-t_0)}, \qquad (t,x(t)) \notin \mathcal{Z}, \tag{57}$$

$$q^T(k,h)E_{\mathrm{d}}^T E_{\mathrm{d}} q(k,h) \leq q^T(h)E_{\mathrm{d}}^T E_{\mathrm{d}} q(h) \\ \cdot \prod_{j=h}^{j-k-1} \Lambda_{\max} A_{\mathrm{d}}^{\mathrm{T}} A_{\mathrm{d}}, \qquad (t,x(t)) \in \mathcal{Z}, \tag{58}$$

we get,

$$q^T(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}} q(t,t_0) \leq v^T(t_0)E_{\mathrm{c}}^T E_{\mathrm{c}} v(t_0) \\ e^{\Lambda_{\max}(M_{\mathrm{c}})(t-t_0)}, \qquad (t,x(t)) \notin \mathcal{Z}, \tag{59}$$

$$q^T(k,h)E_{\mathrm{d}}^T E_{\mathrm{d}} q(k,h) \leq v^T(h)E_{\mathrm{d}}^T E_{\mathrm{d}} v(h) \\ \prod_{j=h}^{j-k-1} \Lambda_{\max} A_{\mathrm{d}}^{\mathrm{T}} A_{\mathrm{d}}, \qquad (t,x(t)) \in \mathcal{Z}, \tag{60}$$

or finally,

$$\|q^T(t,t_0)\|^2_{E_{\mathrm{c}}^T E_{\mathrm{c}}} \leq \|v(t_0)\|^2_{E_{\mathrm{c}}^T E_{\mathrm{c}}} e^{\Lambda_{\max}(M_{\mathrm{c}})(t-t_0)}, \\ (t,x(t)) \notin \mathcal{Z}, \tag{61}$$

$$\|q^T(k,h)\|^2_{E_{\mathrm{d}}^T E_{\mathrm{d}}} \leq \|v(h)\|^2_{E_{\mathrm{d}}^T E_{\mathrm{d}}} \prod_{j=h}^{j-k-1} \Lambda_{\max} A_{\mathrm{d}}^{\mathrm{T}} A_{\mathrm{d}}, \\ (t,x(t)) \in \mathcal{Z}, \tag{62}$$

what we had to prove.

Lemma 3.2. *Suppose that there exist vector $q(t,t_0)$ and $q(k,h)$ defined by,*

$$q(t,t_0) = \Phi(t,t_0)\hat{E}_{\mathrm{c}}\hat{E}_{\mathrm{c}}^{\;D}v(t_0), \qquad (t,x(t)) \notin \mathcal{Z}, \tag{63}$$

$$q(k,h) = \Phi(k,h)\hat{E}_{\mathrm{d}}\hat{E}_{\mathrm{d}}^{\;D}v(h), \qquad (t,x(t)) \in \mathcal{Z}, \tag{64}$$

If,

$$E_{\mathrm{c}}q(t,t_0) = E_{\mathrm{c}}\Phi(t,t_0)\hat{E}_{\mathrm{c}}\hat{E}_{\mathrm{c}}^{\;D}v(t_0), \qquad (t,x(t)) \notin \mathcal{Z}, \tag{65}$$

$$E_{\mathrm{d}}q(k,h) = E_{\mathrm{d}}\Phi(k,h)\hat{E}_{\mathrm{d}}\hat{E}_{\mathrm{d}}^{\;D}v(h), \qquad (t,x(t)) \in \mathcal{Z}, \tag{66}$$

then,

$$\begin{aligned} &\|q(t,t_0)\|^2_{E_{\mathrm{c}}^T E_{\mathrm{c}}} \geq \|v(t_0)\|^2_{E_{\mathrm{c}}^T E_{\mathrm{c}}} e^{\lambda_{\min}(M_{\mathrm{c}})(t-t_0)}, \\ &(t,x(t)) \notin \mathcal{Z}, \end{aligned} \tag{67}$$

$$\begin{aligned} &\|q(k,h)\|^2_{E_{\mathrm{d}}^T E_{\mathrm{d}}} \geq \|v(h)\|^2_{E_{\mathrm{d}}^T E_{\mathrm{d}}} \prod_{j=h}^{j-k-1} \lambda_{\min} A_{\mathrm{d}}^{\mathrm{T}} A_{\mathrm{d}}, \\ &(t,x(t)) \in \mathcal{Z}, \end{aligned} \tag{68}$$

where,

$$\begin{aligned} &\lambda_{\min}(M_{\mathrm{c}}) = \min q^T(t,t_0)M_{\mathrm{c}}q(t,t_0) : \\ &q(t,t_0) \in W_k \backslash 0, q(t,t_0)E_{\mathrm{c}}^T E_{\mathrm{c}}q(t,t_0) = 1, \\ &\qquad (t,x(t)) \notin \mathcal{Z}, \end{aligned} \tag{69}$$

$$\begin{aligned} &\lambda_{\min} A_{\mathrm{d}}^{\mathrm{T}} A_{\mathrm{d}} = \min q^T(k,h)A_{\mathrm{d}}^{\mathrm{T}} A_{\mathrm{d}}q(k,h) : \\ &q(k,h) \in W_k \backslash 0, q(k,h)E_{\mathrm{d}}^T E_{\mathrm{d}}q(k,h) = 1, \\ &\qquad (t,x(t)) \in \mathcal{Z}, \end{aligned} \tag{70}$$

and,

$$M_{\mathrm{c}} = A_{\mathrm{c}}^T E_{\mathrm{c}} + E_{\mathrm{c}}^T A_{\mathrm{c}}, \qquad (t,x(t)) \notin \mathcal{Z}, \tag{71}$$

where,

$$\begin{aligned} &v(t_0) = q(t,t_0), \qquad (t,x(t)) \notin \mathcal{Z}, v(h) = q(k,h), \\ &(t,x(t)) \in \mathcal{Z}, \end{aligned} \tag{72}$$

Proof. Using well known result about minimal eigenvalue
$lambda_{\min}(\cdot)$ from theory of quadratic form, and using the same line of proof as in *Lemma 1*, it is easy to show that *Lemma 2* holds. ☐

4 PRACTICAL STABILITY OF SINGULARLY IMPULSIVE DYNAMICAL SYSTEMS: BELLMAN - GRONWALL APPROACH

In this section we will use result of Bellman - Gronwall lema derived in previous section to derive stability results for the class of singularly impulsive dynamical systems.

First, we give definition of practical stability,

Definition 4.1. *System given with 17 – 20 is* practically stable *with respect to $\{\tau, \alpha, \beta, Q\}$, if and only if there exist $x_0 \in W_k$ that satisfies condition*

$$\|x_0\|^2_Q < \alpha, \tag{73}$$

and,

$$\|x(t)\|^2_Q < \beta, \qquad \forall t \in \tau, \tag{74}$$

Theorem 4.1. *System given with 17 – 20 is* practically stable *with respect to $\{\tau, \alpha, \beta, Q_{\mathrm{c,d}}\}$, if the following conditions are satisfied,*

$$\frac{\gamma_2(Q_{\mathrm{c}})}{\gamma_1(Q_{\mathrm{c}})} < \beta_{rmc}/\alpha_{\mathrm{c}}, \qquad (t,x(t) \notin \mathcal{Z}) \tag{75}$$

$$\begin{aligned} &\Lambda(M_{\mathrm{c}})T + ln\frac{\gamma_2(Q_{\mathrm{c}})}{\gamma_1(Q_{\mathrm{c}})} < ln(\beta_{\mathrm{c}}/\alpha_{\mathrm{c}}), \forall t \in \tau, \\ &\qquad (t,x(t) \notin \mathcal{Z}), \end{aligned} \tag{76}$$

$$\begin{aligned} &\Lambda^k_(A_{\mathrm{d}}^{\mathrm{T}} H_{\mathrm{d}} A_{\mathrm{d}}, Q_{\mathrm{d}}, W_k) \leq \beta_{\mathrm{d}}/\alpha_{\mathrm{d}}), \\ &\qquad \forall k \in \tau, \quad (t,x(t) \in \mathcal{Z}, \end{aligned} \tag{77}$$

where,

$$M_{\rm c} = A_{\rm c}^{\rm T} E_{\rm c} + E_{\rm c}^{\rm T} A_{\rm c}, \qquad (t, x(t)) \notin \mathcal{Z}, \tag{78}$$

and where $\Lambda(\cdot)$ *is defined with (70).*

Proof. Solution of linear singularly impulsive dynamical system given with (17) – (20) is given by,

$$x(t) = \Phi(t, t_0)\hat{E}_{\rm c}\hat{E}_{\rm c}^{\ D} x_0, \qquad (t, x(t)) \notin \mathcal{Z}, \tag{79}$$

$$x(k) = \Phi(k, h)\hat{E}_{\rm d}\hat{E}_{\rm d}^{\ D} x_0, \quad k > 1, \quad (t, x(t)) \in \mathcal{Z}, \tag{80}$$

and multiplying the equation (81) with $E_{\rm c,d}$ from the left side,

$$E_{\rm c} x(t) = E_{\rm c}\Phi(t, t_0)\hat{E}_{\rm c}\hat{E}_{\rm c}^{\ D} x_0, \qquad (t, x(t)) \notin \mathcal{Z}, \tag{81}$$

$$E_{\rm d} x(k) = E_{\rm d}\Phi(k, h)\hat{E}_{\rm d}\hat{E}_{\rm d}^{\ D} x_0, \qquad (t, x(t)) \in \mathcal{Z}, \tag{82}$$

By applying Lemma 1,

$$\|x(t, t_0)\|^2_{E_{\rm c}^T E_{\rm c}} \leq \|x(t_0)\|^2_{E_{\rm c}^T E_{\rm c}} e^{\Lambda_{\max}(M_{\rm c})(t-t_0)}, \quad (t, x(t)) \notin \mathcal{Z}, \tag{83}$$

$$\|x(k, h)\|^2_{E_{\rm d}^T E_{\rm d}} \leq \|x(h)\|^2_{E_{\rm d}^T E_{\rm d}} \Lambda^k_{\max} A_{\rm d}^{\rm T} A_{\rm d}, \quad (t, x(t)) \in \mathcal{Z}, \tag{84}$$

by using the condition 73 in Definition 1,

$$\|x(t, t_0)\|^2_{E_{\rm c}^T E_{\rm c}} \leq \alpha e^{\Lambda_{\max}(M_{\rm c})(t-t_0)}, \qquad (t, x(t)) \notin \mathcal{Z}, \tag{85}$$

$$\|x(k, h)\|^2_{E_{\rm d}^T E_{\rm d}} \leq \alpha \Lambda^k_{\max} A_{\rm d}^{\rm T} A_{\rm d}, \qquad (t, x(t)) \in \mathcal{Z}, \tag{86}$$

and by using the basic condition 75 with $Q = I$, where I is identity matrix, of Theorem 1,

$$\|x(t, t_0)\|^2_{E_{\rm c}^T E_{\rm c}} \leq \alpha e^{\Lambda_{\max}(M_{\rm c})(t-t_0)} < \alpha \frac{\beta}{\alpha}, \qquad \forall t \in \tau, \quad (t, x(t)) \notin \mathcal{Z}, \tag{87}$$

$$\|x(k, h)\|^2_{E_{\rm d}^T E_{\rm d}} \leq \alpha \Lambda^k_{\max} A_{\rm d}^{\rm T} A_{\rm d} < \alpha \frac{\beta}{\alpha}, \qquad \forall t \in \tau, \quad (t, x(t)) \in \mathcal{Z}, \tag{88}$$

i.e.,

$$\|x(t, t_0)\|^2_{E_{\rm c}^T E_{\rm c}} < \beta, \qquad \forall t \in \tau, \qquad (t, x(t)) \notin \mathcal{Z}, \tag{89}$$

$$\|x(k, h)\|^2_{E_{\rm d}^T E_{\rm d}} < \beta, \qquad \forall t \in \tau, \qquad (t, x(t)) \in \mathcal{Z}, \tag{90}$$

what has to be proved. □

These results can be further extended for the class of singularly impulsive dynamical system under the input or perturbing forces, and these resulkts are under the development.

5 CONCLUSION

In this paper for the class of singularly impulsive dynamical systems we have derived Bellman - Gronwall Lemma. Further, we have applied this result to prove practical stability for the class of singularly impulsive dynamical system. We have derived results for the class of linear system in free regime and results under the perturbing forces are under the development.

6 FURTHER RESEARCH

These results will be further extended to the singularly impulsive dynamical systems with time delay.

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