The 6th International Federation of Nonlinear Analysts Conference (2012) 67–76

Generalized Inverses for a Special Structure Matrix Appearing in Control

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ABSTRACT: In the literature of control and system theory, several explicit formulae for solving structural systems and computing the inverse of those are appeared regularly. Recently, getting inspired from an interesting application of control for the change of the initial state of a linear system of higher order in (almost) zero time, the need for the input calculation, and the matrix properties of different types of generalized inverses for a special structure matrix, like

$$\begin{bmatrix} 1 & \mu & \mu^2 & \mu^3 & * & \cdots & * & * & \cdots & \mu^{n-1} \\ 1 & \lambda & \lambda^2 & \lambda^3 & * & \cdots & * & * & \cdots & \lambda^{n-1} \\ 0 & 1 & 2\lambda & 3\lambda^2 & * & \cdots & * & * & \cdots & (n-1)\lambda^{n-2} \\ 0 & 0 & 1 & 3\lambda & * & \cdots & * & * & \cdots & (n-1)(n-2)\lambda^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & * & \cdots & \frac{1}{(m-1)}\frac{d^{m-1}}{d\lambda_j^{m-1}}\left(\lambda^{n-1}\right) \end{bmatrix}$$

where $\lambda \neq \mu \neq 0$ have been derived, studied and considered further in the present paper. The above matrix is appeared for the very first time in the literature of matrix theory, according to the authors' knowledge. Two numerical examples are also provided.

Keywords: Linear System; Special Structure Matrix; LU Parameterization

1 INTRODUCTION

In the literature of control and system theory, the idea of approximating distributional inputs with smooth functions that achieve similar (control and system) objectives was first introduced by Gupta and Hasdorff, see [2] and [3]. Quite recently, the transfer of the initial state of an open loop, linear higher-order descriptor (regular) differential system in (practically speaking, almost) zero-time has been fully investigated, see [4], i.e.

$$F\underline{x}^{(r)}(t) = G\underline{x}(t) + \underline{b}u(t),$$

with known initial conditions $\underline{x}(t_o)$, $\underline{x}'(t_o)$, ..., $\underline{x}^{(r-1)}(t_o)$,
where $F, G \in \mathcal{M}(n \times n; \mathbb{F})$, and $\underline{b} \in \mathcal{M}(n \times 1; \mathbb{F})$ (i.e. $\mathcal{M}$ is the algebra of $n \times m$ matrices with elements in the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with $\det F = 0$ (0 is the zero element of $\mathcal{M}(n = 1, \mathbb{F})$), $\underline{x}(t) \in \mathcal{C}^\infty(\mathbb{F}, \mathcal{M}(n \times 1; \mathbb{F}))$ and $u(t) \in \mathcal{C}^\infty(\mathbb{F}, \mathcal{M}(1 \times 1; \mathbb{F}))$. For the sake of simplicity, we set in the sequel $\mathcal{M}_n \triangleq \mathcal{M}(n \times n; \mathbb{F})$ and $\mathcal{M}_{n,m} \triangleq \mathcal{M}(n \times m; \mathbb{F})$.

In order this problem to have a solution, the appropriate input has to be made up as a linear combination of the δ-function of Dirac and its derivatives, for more details see [2]-[4] and [5] and references theirin, i.e.

$$u_o(t) = \sum_{k=1}^{n-1} a_k \delta^{(k)}(t), \tag{1}$$

where $\delta^{(k)}(t)$ or $\frac{d^k \delta(t)}{dt^k}$ is the k^{th}-derivative of the Dirac δ-function, and a_i for $i = 0, 1, \ldots, n-1$ are the magnitudes of the delta function and its derivatives.

Furthermore, we assume that the state of the system at time 0^- is

$$\underline{x}(0^-) = \underline{x}'(0^-) = \cdots = \underline{x}^{(r-1)}(0^-) = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}^t,$$

and at time 0^+, it achieves

$$\underline{x}(0^+) = \begin{bmatrix} x_1^0 & x_2^0 & \ldots & x_n^0 \end{bmatrix}^t, \underline{x}'(0^+) = \begin{bmatrix} x_1^1 & x_2^1 & \ldots & x_n^1 \end{bmatrix}^t, \ldots,$$
$$\underline{x}^{(r-1)}(0^+) = \begin{bmatrix} x_1^{r-1} & x_2^{r-1} & \ldots & x_n^{r-1} \end{bmatrix}^t.$$

Received August 10, 2012 IFNA Publishers, Inc.

In paper [4], similarly as in [2], [3] and [5], a classical approximated expression for the controller (1), which is based on the Gaussian (Normal) function, is used. Thus, by considering what is Dirac δ-function and the Gaussian (Normal) function, we obtain

$$\delta\left(t\right)=\lim_{\sigma\to 0}\frac{1}{\sigma\sqrt{2\pi}}e^{-t^2/2\sigma^2}=\lim_{\sigma\to 0}\frac{1}{\sigma}\Phi\left(\frac{t}{\sigma}\right),$$

where $\Phi\left(x\right)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

So, the approximate expression for the impulsive-input (1) is given by

$$u\left(t\right)=\sum_{k=0}^{n-1}\frac{1}{\sigma^{k+1}}\Phi^{(k)}\left(\frac{t}{\sigma}\right)a_k.$$

Then, we can take the limit $u_o\left(t\right)=\lim_{\sigma\to 0}\ u\left(t\right)$.

Thus, the unknown vector-coefficient $\underline{a}=\begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1}\end{bmatrix}^t$, where $a_i\in\mathbb{F}$ for $i=0,1,\ldots,n-1$ has been analytically calculated by solving the system (2).

$$\begin{bmatrix} V_l \\ V_{(l+1)_{z_{l+1}}} \\ V_{(l+2)_{z_{l+2}}} \\ \vdots \\ V_{\kappa_{d_\kappa}} \end{bmatrix}\underline{a}=\begin{bmatrix} \underline{z}_l\left(0^+\right) \\ \underline{z}_{(l+1)_{z_{l+1}}}\left(0^+\right) \\ \underline{z}_{(l+2)_{z2}}\left(0^+\right) \\ \vdots \\ \underline{z}_{\kappa_{z_\kappa}}\left(0^+\right) \end{bmatrix}, \tag{2}$$

where the vector $\begin{bmatrix} \underline{z}_l^t\left(0^+\right) & \underline{z}_{(l+1)_{z_{l+1}}}^t\left(0^+\right) & \underline{z}_{(l+2)_{z2}}^t\left(0^+\right) & \cdots & \underline{z}_{\kappa_{z_\kappa}}^t\left(0^+\right)\end{bmatrix}^t$ is constant, $V_l\in\mathcal{M}_{s,n}$ is a *rectangular* $s\times n$*-Vandermonde* matrix and $V_{j_{z_j}}\in\mathcal{M}_{\mu_{z_j},n}$, with $j=l+1,\ l+2,\ldots,\ \kappa$ and $z_j=1,2,\ldots,d_j$, is a special structure matrix.

Obviously, the system (2) can be further transposed into a more convenient system.

Analytically, if we multiply the 1^{st} row of the Vandermonde matrix V_l, i.e. $\begin{bmatrix} 1 & \lambda & \lambda^2 & \ldots & \lambda^{n-1}\end{bmatrix}$, with the number (-1) and we added it to the 1^{st} row of each of $V_{(l+1)_{z_{l+1}}}$, $V_{(l+2)_{z_{l+2}}},\ldots,\ V_{\kappa_{z_\kappa}}$, then $V_{j_{z_j}}$ is given by

$$V_{j_{z_j}}=\begin{bmatrix} 0 & \mu_j-\lambda & \mu_j^2-\lambda^2 & \cdots & \cdots & \multicolumn{3}{c}{\mu_j^{n-1}-\lambda^{n-1}} \\ 0 & 1 & 2\mu_j & \cdots & \cdots & \multicolumn{3}{c}{(n-1)\mu_j^{n-2}} \\ 0 & 0 & 1 & \cdots & \cdots & \multicolumn{3}{c}{\left(n-1\right)\left(n-2\right)\mu_j^{n-3}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \multicolumn{3}{c}{\vdots} \\ 0 & 0 & \ldots & 0 & 1 & \frac{1}{(\rho_j-1)!}\frac{d^{\rho_j-1}\mu_j^{\rho_j+1}}{d\mu_j^{\rho_j-1}} & \cdots & \frac{1}{(\rho_j-1)!}\frac{d^{\rho_j-1}\mu_j^{n-1}}{d\mu_j^{\rho_j-1}} \end{bmatrix} \tag{3}$$

Then, we can easily observe that the 1^{st} row of the matrix (3) can be re-written as below, i.e. the element $\mu_j^\kappa+\mu_j^{\kappa-1}\lambda+\ldots+\mu_j\lambda^{\kappa-1}+\lambda^\kappa=\sum\limits_{\substack{k_1,k_2=0\\ \sum_{i=1}^{2}k_i=\kappa}}^{\kappa}\mu_j^{k_1}\lambda^{k_2}$.

So, the first row is presented as

$$\left(\mu_j-\lambda\right)\begin{bmatrix} 0 & 1 & \sum\limits_{\substack{k_1,\ k_2=0\\ \sum_{i=1}^{2}k_i=1}}^{1}\mu_j^{k_1}\lambda^{k_2} & \sum\limits_{\substack{k_1,\ k_2=0\\ \sum_{i=1}^{2}k_i=2}}^{2}\mu_j^{k_1}\lambda^{k_2} & \ldots & \sum\limits_{\substack{k_1,\ k_2=0\\ \sum_{i=1}^{2}k_i=n-2}}^{n-2}\mu_j^{k_1}\lambda^{k_2}\end{bmatrix}.$$

Now, since, the element $\mu_j-\lambda\neq 0$, we can multiply by left the eq. (3) with a properly chosen transformation matrix, so as to obtain

$$S_{j_{z_j}}\triangleq\begin{bmatrix} 0 & 1 & \sum\limits_{\substack{k_1,k_2=0\\ \sum_{i=1}^{2}k_i=1}}^{1}\mu_j^{k_1}\lambda^{k_2} & \cdots & \cdots & \cdots & \multicolumn{2}{c}{\sum\limits_{\substack{k_1,k_2=0\\ \sum_{i=1}^{2}k_i=n-2}}^{n-2}\mu_j^{k_1}\lambda^{k_2}} \\ 0 & 1 & 2\mu_j & \cdots & \cdots & \cdots & \multicolumn{2}{c}{(n-1)\mu_j^{n-2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \multicolumn{2}{c}{\vdots} \\ 0 & \cdots & \cdots & 1 & \frac{1}{(\rho_j-1)!}\frac{d^{\rho_j-1}\mu_j^{\rho_j+1}}{d\mu_j^{\rho_j-1}} & \cdots & \frac{1}{(\rho_j-1)!}\frac{d^{\rho_j-1}\mu_j^{n-1}}{d\mu_j^{\rho_j-1}} \end{bmatrix}. \tag{4}$$

Finally, the system (5) is obtained, where the matrices $S_{j_{z_j}}$ for $j = l+1, l+2, \ldots, \kappa$ are derived by taking into account a properly chosen transformation left-matrix Z, as follows

$$Z\begin{bmatrix} V_l \\ V_{(l+1)_{z_{l+1}}} \\ V_{(l+2)_{z_{l+2}}} \\ \vdots \\ V_{\kappa_{d_\kappa}} \end{bmatrix}\underline{a} = Z\begin{bmatrix} \underline{z}_l^*\left(0^+\right) \\ \underline{z}_{(l+1)_{z_{l+1}}}\left(0^+\right) \\ \underline{z}_{(l+2)_{z2}}\left(0^+\right) \\ \vdots \\ \underline{z}_{\kappa_{z_\kappa}}\left(0^+\right) \end{bmatrix} \Leftrightarrow \begin{bmatrix} V_l \\ S_{l+1} \\ S_{l+2} \\ \vdots \\ S_\kappa \end{bmatrix}\underline{a} = \begin{bmatrix} \underline{z}_l\left(0^+\right) \\ \underline{d}_{l+1}\left(0^+\right) \\ \underline{d}_{l+2}\left(0^+\right) \\ \vdots \\ \underline{d}_\kappa\left(0^+\right) \end{bmatrix}, \tag{5}$$

where $V_l \in \mathcal{M}_{l,n}$, $S_j \in \mathcal{M}_{j,n}$, $\underline{z}_l\left(0^+\right) \in \mathcal{M}_{l,1}$ and $\underline{d}_{l+2}\left(0^+\right) \in \mathcal{M}_{j,1}$, for $j = l+1, l+2, \ldots, \kappa$.
Note that $\rho_j = \max\limits_{z_i=1,2,\ldots,d_j} \mu_{z_j}$ is the index of annihilation for the eigenvalue μ_{z_j}.

Consequently, the system (5) contains the following sub-systems.

$$\left.\begin{array}{c} V_l\underline{a} = \underline{z}_l\left(0^+\right) \\ S_{l+1}\underline{a} = \underline{d}_{l+1}\left(0^+\right) \\ \cdots \\ S_\kappa\underline{a} = \underline{d}_\kappa\left(0^+\right) \end{array}\right\},$$

where V_l, S_j for $j = l+1, l+2, \ldots, \kappa$ are non-square matrices.

Inevitably for the analytic solution of the above system, and consequently for the determination of the coefficients $\underline{a}$ of the input (1), some elements of the generalized inverse theory are needed.

Thereafter, before we go further describing the main results of the present paper; see the next section, it sould be mentioned that the results for the rectangular Vandermonde matrix are presented in the paper [6]. So based on [6], here in the next section, we give the closed form of the $\{1,2,3\}$- generalized inverse of a very special rectangular matrix S. This matrix is derived naturally from the above mentioned control problem for higher order systems, and it is appeared for the very first time in the literature of matrix theory according to the authors' knowledge. Additionally, for the better understanding of the presented results, some numerical examples are also considered. The 3^{rd} section concludes the whole paper.

As a last part of this introduction, the following basic definitions for different kind of generalized inverses are simply repeated; see for more details [1].

Definition 1.1. *Denote the square matrix $A \in \mathcal{M}_n$. We say that the non-negative integer k is the index of A, $Ind\left(A\right) = k$, if k is the smallest non-negative integer such as*

$$rank\left(A^k\right) = rank\left(A^{k+1}\right).$$

Definition 1.2. *The Moore-Penrose inverse of a rectangular matrix $A \in \mathcal{M}_{m,n}$ is the matrix $A^\dagger \in \mathcal{M}_{m,n}$ such that*

(1) $AA^\dagger A = A$,
(2) $A^\dagger AA^\dagger = A^\dagger$,
(3) $\left(AA^\dagger\right)^* = AA^\dagger$,
(4) $\left(A^\dagger A\right)^* = A^\dagger A$,

where $*$ the conjugate transpose index of the relevant matrix.

Moreover, the *Drazin inverse* of square matrix $A \in \mathcal{M}_n$, $Ind\left(A\right) = k$ is the matrix A^D satisfying

(i) $A^DAA^D = A^D$,
(ii) $AA^D = A^DA$,
(iii) $A^{l+1}A^D = A^l$,

for $l \geqslant k = Ind\left(A\right)$.
Note that if A is non-singular, then $A^\dagger \equiv A^D \equiv A^{-1}$.

2 MAIN RESULTS

Before, we present the main results of this paper, the following definitions are needed.

Definition 2.1. *Consider the following matrices:*
a) Let $P_i(a)$ be a $m \times m$-matrix which has a non-zero element a in the i^{th}-row and the j^{th}-column, i.e.

$$P_i(a) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & \mathbb{O} & & \\ & & 1 & & & & \\ & & & a & & & \\ & & & & 1 & & \\ & \mathbb{O} & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}. \tag{6}$$

Thus, whenever a matrix A is multiplied from the left by $P_i(a)$ then the i^{th}-row of it is multiplied by the non-zero number a.
b) Let $P_i(j,a)$ be a $m \times m$-matrix which has a non-zero element a in the i^{th}-row and the j^{th}-column, i.e.

$$P_i(j,a) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & \mathbb{O} & & \\ & & 1 & \cdots & a & & \\ & & & \ddots & \vdots & & \\ & & & & 1 & & \\ & \mathbb{O} & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}. \tag{7}$$

Thus, whenever a matrix A is multiplied from the left by $P_i(j,a)$ then the j^{th}-row of it is multiplied by the non-zero number a and it is added to the j^{th}-row of A.
c) Let $Q_i(a)$ be a $n \times n$-matrix which has a non-zero element a in the j^{th}-row and the i^{th}-column, i.e.

$$Q_i(a) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & \mathbb{O} & & \\ & & 1 & & & & \\ & & & a & & & \\ & & & & 1 & & \\ & \mathbb{O} & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \tag{8}$$

Thus, whenever a matrix A is multiplied from the right by $Q_i(a)$ then the i^{th}-column of it is multiplied by the non-zero number a.
d) Let $Q_i(j,a)$ be a $n \times n$-matrix which has a non-zero element a in the j^{th}-row and the i^{th}-column, i.e.

$$Q_i(j,a) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & \mathbb{O} & & \\ & & 1 & & & & \\ & & \vdots & \ddots & & & \\ & & a & \cdots & 1 & & \\ & \mathbb{O} & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}. \tag{9}$$

Thus, whenever a matrix A is multiplied from the right by $Q_i(j,a)$ then the i^{th}-column of it is multiplied by the non-zero number a and it is added to the j^{th}-column of A.

Definition 2.2. *Let us define with the $\coprod \cdot$ symbol the order left multiplication of matrices as it is given by $\coprod_{m}^{j=1} P_j = P_m P_{m-1} \cdots P_2 P_1$.*

As we have already discussed extensively in the introduction, in an interesting recent applications of the control and system theory, see [4], we need to calculate the generalized inverses of a very special matrix, like

$$\begin{bmatrix} 1 & \mu & \mu^2 & \mu^3 & * & \cdots & * & * & \cdots & \mu^{n-1} \\ 1 & \lambda & \lambda^2 & \lambda^3 & * & \cdots & * & * & \cdots & \lambda^{n-1} \\ 0 & 1 & 2\lambda & 3\lambda^2 & * & \cdots & * & * & \cdots & (n-1)\,\lambda^{n-2} \\ 0 & 0 & 1 & 3\lambda & * & \cdots & * & * & \cdots & (n-1)\,(n-2)\,\lambda^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & * & \cdots & \frac{1}{(m-1)}\frac{d^{m-1}}{d\lambda_j^{m-1}}\left(\lambda^{n-1}\right) \end{bmatrix} \in \mathcal{M}_{m+1,n},$$

where $\lambda \neq \mu \neq 0$.

In this section, we investigate the rectangular matrix, where $n > m$, using the first row of the Vandermonde matrix, see also introduction. The other two cases (where $n < m$ and $n = m$) can be straightforwardly derived using similar arguments, and following ideas presented in [6]. So, let assume that we want to investigate the following matrix.

$$S_{m,n} = \begin{bmatrix} 0 & \lambda-\mu & \lambda^2-\mu^2 & \lambda^3-\mu^3 & * & \cdots & * & * & \cdots & \lambda^{n-1}-\mu^{n-1} \\ 0 & 1 & 2\lambda & 3\lambda^2 & * & \cdots & * & * & \cdots & (n-1)\,\lambda^{n-2} \\ 0 & 0 & 1 & 3\lambda & * & \cdots & * & * & \cdots & (n-1)\,(n-2)\,\lambda^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & * & \cdots & \frac{1}{(m-1)}\frac{d^{m-1}}{d\lambda_j^{m-1}}\left(\lambda^{n-1}\right) \end{bmatrix} \in \mathcal{M}_{m,n}. \quad (10)$$

Consequently, we transport the rectangular special matrix (10) into the following form,

$$\left[\begin{array}{ccc} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \end{array}\right].$$

Proposition 2.1. ***(Special Matrix parameterization)***
There are invertible matrices $\mathbf{P} \in \mathcal{M}_m$ and $\mathbf{Q} \in \mathcal{M}_n$ such that

$$\mathbf{P} S_{m,n} \mathbf{Q} = \left[\begin{array}{ccc} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \end{array}\right], \quad (11)$$

where the permutated matrices are given analytically by the following expressions (12) and (13), i.e.

$$\mathbf{P} = \prod_{m}^{s=2} P_{m-s+2}\left(\frac{1}{\lambda-\mu}\right) P_{m-s+2}\,(m-s+1,-1), \quad (12)$$

where $P_i\,(a)$ and $P_i\,(j,a)$ are given by (6) and (7) respectively, and

$$\mathbf{Q} = \prod_{s=2}^{m+1}\prod_{k=s+1}^{n} Q_k\left(s, -\frac{1}{(s-2)!}\frac{d^{s-2}}{d\lambda^{s-2}}\left(\sum_{\substack{k_1,k_2=0\\k_1+k_2=k-2}}^{k-2}\mu^{k_1}\lambda^{k_2}\right)\right), \quad (13)$$

where $Q_i\,(j,a)$ is given by (9).

Proof. We start with the matrix (10) where $\lambda \neq \mu \neq 0$.
In this direction, we work as follows

$$\begin{gathered} P_2\,(1,-1)\,S_{m,n} \to P_2\left(\frac{1}{\lambda-\mu}\right) P_2\,(1,-1)\,S_{m,n} \to P_3\,(2,-1)\,P_2\left(\frac{1}{\lambda-\mu}\right) P_2\,(1,-1)\,S_{m,n} \\ \to P_3\left(\frac{1}{\lambda-\mu}\right) P_3\,(2,-1)\,P_2\left(\frac{1}{\lambda-\mu}\right) P_2\,(1,-1)\,S_{m,n} \to \cdots \to \\ P_m\left(\frac{1}{\lambda-\mu}\right) P_m\,(m-1,-1)\,P_{m-1}\left(\frac{1}{\lambda-\mu}\right) P_{m-1}\,(m-2,-1)\cdots \\ P_3\left(\frac{1}{\lambda-\mu}\right) P_3\,(2,-1)\,P_2\left(\frac{1}{\lambda-\mu}\right) P_2\,(1,-1)\,S_{m,n}. \end{gathered}$$

So, the matrix (10) is transformed to

$$\begin{bmatrix} 0 & 1 & \mu+\lambda & \mu^2+\mu\lambda+\lambda^2 & * & \cdots & * & * & \cdots & \sum\limits_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2} \\ 0 & 0 & 1 & \frac{d}{d\lambda}\left(\mu^2+\mu\lambda+\lambda^2\right) & * & \cdots & * & * & \cdots & \frac{d}{d\lambda}\sum\limits_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & * & \cdots & \frac{1}{(m-1)!}\frac{d^{m-1}}{d\lambda^{m-1}}\sum\limits_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2} \end{bmatrix}$$

Thus, we conclude to the determination of matrix $\mathbf{P}$

$$\begin{aligned}\mathbf{P} &\triangleq P_m\left(\frac{1}{\lambda-\mu}\right)P_m\left(m-1,-1\right)\cdots P_2\left(\frac{1}{\lambda-\mu}\right)P_2\left(1,-1\right)\\ &=\coprod_{m}^{s=2}P_{m-s+2}\left(\frac{1}{\lambda-\mu}\right)P_{m-s+2}\left(m-s+1,-1\right).\end{aligned}$$

Now we want to transfer the $\mathbf{P}S_{m,n}$ into the desired matrix $\begin{bmatrix}\underline{0}_m & I_m & \mathbb{O}_{m,n-m-1}\end{bmatrix}$.
So, we act as follows

$$\mathbf{P}S_{m,n}Q_3\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{1}\mu^{k_1}\lambda^{k_2}\right)\to\mathbf{P}S_{m,n}Q_3\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{1}\mu^{k_1}\lambda^{k_2}\right)Q_4\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=2}}^{2}\mu^{k_1}\lambda^{k_2}\right)$$

$$\to\ldots\to$$

$$\mathbf{P}S_{m,n}Q_3\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{1}\mu^{k_1}\lambda^{k_2}\right)Q_4\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=2}}^{2}\mu^{k_1}\lambda^{k_2}\right)\cdots Q_n\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)\to$$

$$\mathbf{P}S_{m,n}Q_3\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{1}\mu^{k_1}\lambda^{k_2}\right)Q_4\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=2}}^{2}\mu^{k_1}\lambda^{k_2}\right)\cdots Q_n\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)\cdot$$

$$\cdot Q_3\left(3,-\frac{d}{d\lambda}\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{2}\mu^{k_1}\lambda^{k_2}\right)\to\cdots\to$$

$$\mathbf{P}S_{m,n}Q_3\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{1}\mu^{k_1}\lambda^{k_2}\right)\cdots Q_n\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)Q_4\left(3,-\frac{d}{d\lambda}\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{2}\mu^{k_1}\lambda^{k_2}\right)$$

$$\cdots Q_n\left(3,-\frac{d}{d\lambda}\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)$$

$$\to\ldots\to$$

$$\mathbf{P}S_{m,n}Q_3\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{1}\mu^{k_1}\lambda^{k_2}\right)\cdots Q_n\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)Q_4\left(3,-\frac{d}{d\lambda}\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{2}\mu^{k_1}\lambda^{k_2}\right)$$

$$\cdots Q_n\left(3,-\frac{d}{d\lambda}\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)$$

$$\cdots Q_5\left(4,-\frac{1}{2!}\frac{d^2}{d\lambda^2}\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{3}\mu^{k_1}\lambda^{k_2}\right)\cdots Q_n\left(2,-\frac{1}{2!}\frac{d^2}{d\lambda^2}\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)\cdots$$

$$Q_{m+2}\left(m+1,-\frac{1}{(\rho_j-1)!}\frac{d^{m-1}}{d\lambda^{m-1}}\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{m}\mu^{k_1}\lambda^{k_2}\right)\cdots Q_n\left(m+1,-\frac{1}{(m-1)!}\frac{d^{m-1}}{d\lambda^{m-1}}\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)$$

We also define matrix

$$Q \triangleq Q_3\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{1}\mu^{k_1}\lambda^{k_2}\right)\cdots Q_n\left(2,-\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)Q_4\left(3,-\frac{d}{d\lambda}\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{2}\mu^{k_1}\lambda^{k_2}\right)\cdots$$

$$Q_n\left(3,-\frac{d}{d\lambda}\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)\cdots Q_5\left(4,-\frac{1}{2!}\frac{d^2}{d\lambda^2}\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{3}\mu^{k_1}\lambda^{k_2}\right)\cdots Q_n\left(2,-\frac{1}{2!}\frac{d^2}{d\lambda^2}\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)\cdots$$

$$Q_{m+2}\left(m+1,-\frac{1}{(\rho_j-1)!}\frac{d^{m-1}}{d\lambda^{m-1}}\sum_{\substack{k_1,k_2=0\\k_1+k_2=1}}^{m}\mu^{k_1}\lambda^{k_2}\right)\cdots Q_n\left(m+1,-\frac{1}{(m-1)!}\frac{d^{m-1}}{d\lambda^{m-1}}\sum_{\substack{k_1,k_2=0\\k_1+k_2=n-2}}^{n-2}\mu^{k_1}\lambda^{k_2}\right)$$

$$=\prod_{s=2}^{m+1}\prod_{k=s+1}^{n}P_k\left(s,-\frac{1}{(s-2)!}\frac{d^{s-2}}{d\lambda^{s-2}}\left(\sum_{\substack{k_1,k_2=0\\k_1+k_2=k-2}}^{k-2}\mu^{k_1}\lambda^{k_2}\right)\right)$$

Consequently, we have transposed the special matrix (10) into $\begin{bmatrix}\underline{0}_m & I_m & \mathbb{O}_{m,n-m-1}\end{bmatrix}$. □

Example 2.1. *Suppose that we have the* 3×4*-special matrix,*

$$S_{3,4}=\begin{bmatrix}0 & 10-3 & 10^2-3^2 & 10^3-3^3\\ 0 & 1 & 2\cdot 10 & 3\cdot 10^3\\ 0 & 0 & 1 & 3\cdot 10^3\end{bmatrix}\in\mathcal{M}_{3,4},$$

then by applying (12), we take

$$\mathbf{P}=\coprod_{3}^{s=2}P_{3-s+2}\left(\frac{1}{7}\right)P_{3-s+2}\left(3-s+1,-1\right)=P_3\left(\frac{1}{7}\right)P_3\left(2,-1\right)P_2\left(\frac{1}{7}\right)P_2\left(1,-1\right)$$
$$=\begin{bmatrix}1 & 0 & 0\\ -1/7 & 1/7 & 0\\ 1/49 & -1/49 & 1/7\end{bmatrix}.$$

The matrix $S_{3,4}$ is being transformed into the following

$$\mathbf{P}S_{3,4}=\begin{bmatrix}0 & 1 & \mu+\lambda & \mu^2+\mu\lambda+\lambda^2\\ 0 & 0 & 1 & \mu+2\lambda\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 3+10 & 3^2+3\cdot 10+10^2\\ 0 & 0 & 1 & 3+2\cdot 10\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 13 & 139\\ 0 & 0 & 1 & 23\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Now, we want to transfer the $\mathbf{P}S_{3,4}$ into the matrix $\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$*, so we apply (13), i.e.*

$$\mathbf{Q}=\prod_{s=2}^{4}\prod_{k=s+1}^{4}Q_k\left(s,-\frac{1}{(s-2)!}\frac{d^{s-2}}{d\lambda^{s-2}}\left(\sum_{\substack{k_1,k_2=0\\k_1+k_2=k-2}}^{k-2}\mu^{k_1}\lambda^{k_2}\right)\right)=Q_3\left(2,-13\right)Q_3\left(2,-139\right)Q_4\left(3,-23\right)$$
$$=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -13 & 160\\ 0 & 0 & 1 & -23\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

So, we take the parameterization

$$\mathbf{P}S_{3,4}\mathbf{Q}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

where $\mathbf{P}=\begin{bmatrix}1 & 0 & 0\\ -1/7 & 1/7 & 0\\ 1/49 & -1/49 & 1/7\end{bmatrix}$ *and* $\mathbf{Q}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -13 & 160\\ 0 & 0 & 1 & -23\\ 0 & 0 & 0 & 1\end{bmatrix}$.

In the next lines, the generalized inverse of the rectangular special matrix (10) is derived.

Theorem 2.1. *The $\{1,2,3\}$-inverse of the rectangular special matrix (10) is given by*

$$S_{n,m}^{\{1,2,3\}} = \mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P},$$

where the permutated matrices $\mathbf{P}$ and $\mathbf{Q}$ are given by the expressions (12) and (13), respectively.

Proof. Consider the expression (11), i.e.

$$\mathbf{P}S_{m,n}\mathbf{Q} = \begin{bmatrix} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \end{bmatrix} \Leftrightarrow S_{m,n} = \mathbf{P}^{-1}\begin{bmatrix} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \end{bmatrix}\mathbf{Q}^{-1}.$$

In order the matrix $\mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P}$ to be the $1,2,3$-inverse of $S_{m,n}$, we have to prove the following three equalities,

(1) $S_{m,n}\mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P}S_{m,n} = S_{m,n}$,

(2) $\mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P}S_{m,n}\mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P} = \mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P}$

and

(3) $\left(S_{m,n}\mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P}\right)^* = S_{m,n}\mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P}$.

Thus, the (1) holds since

$$\begin{aligned}
&S_{m,n}\mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P}S_{m,n} \\
&= \mathbf{P}^{-1}\begin{bmatrix} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \end{bmatrix}\mathbf{Q}^{-1}\mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P}\mathbf{P}^{-1}\begin{bmatrix} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \end{bmatrix}\mathbf{Q}^{-1} \\
&= \mathbf{P}^{-1}\begin{bmatrix} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \end{bmatrix}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\begin{bmatrix} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \end{bmatrix}\mathbf{Q}^{-1} \\
&= \mathbf{P}^{-1}\left(\mathbb{O}_m + I_m + \mathbb{O}_m\right)\begin{bmatrix} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \end{bmatrix}\mathbf{Q}^{-1} \\
&= S_{m,n},
\end{aligned}$$

and the (2) holds since

$$\begin{aligned}
&\mathbf{Q}\begin{bmatrix} \underline{0}^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P}S_{m,n}\mathbf{Q}\begin{bmatrix} \underline{0}^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P} \\
&= \mathbf{Q}\begin{bmatrix} \underline{0}^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P}\mathbf{P}^{-1}\begin{bmatrix} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \end{bmatrix}\mathbf{Q}^{-1}\mathbf{Q}\begin{bmatrix} \underline{0}^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P} \\
&= \mathbf{Q}\begin{bmatrix} \underline{0}^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\begin{bmatrix} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \end{bmatrix}\begin{bmatrix} \underline{0}^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P} \\
&= \mathbf{Q}\begin{bmatrix} 0 & \underline{0}_m^t & \underline{0}_{n-m-1}^t \\ \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1} \\ \underline{0}_{n-m-1} & \mathbb{O}_{n-m-1,m} & \mathbb{O}_{n-m-1} \end{bmatrix}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P} \\
&= \mathbf{Q}_1\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m} \end{bmatrix}\mathbf{P}_1,
\end{aligned}$$

and finally, the (3) also holds since

$$\left(S_{m,n}\mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m}\end{bmatrix}\mathbf{P}\right)^* = \left(\mathbf{P}^{-1}\begin{bmatrix} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1}\end{bmatrix}\mathbf{Q}^{-1}\mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m}\end{bmatrix}\mathbf{P}\right)^*$$

$$= \left(\mathbf{P}^{-1}\begin{bmatrix} \underline{0}_m & I_m & \mathbb{O}_{m,n-m-1}\end{bmatrix}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m}\end{bmatrix}\mathbf{P}\right)^*$$

$$= \left(\mathbf{P}^{-1}\left(\mathbb{O}_m + I_m + \mathbb{O}_m\right)\mathbf{P}\right)^*$$

$$= I_m = S_{m,n}\mathbf{Q}\begin{bmatrix} \underline{0}_m^t \\ I_m \\ \mathbb{O}_{n-m-1,m}\end{bmatrix}\mathbf{P}.$$

□

Example 2.2. *Suppose that we have the 3×4- special matrix of Example 2.1, then the $1,2,3$-inverse of the special matrix is given by*

$$S_{4,3}^{\{1,2,3\}} = \mathbf{Q}\begin{bmatrix} \underline{0}_3^t \\ I_3 \end{bmatrix}\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -13 & 160 \\ 0 & 0 & 1 & -23 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \underline{0}_3^t \\ I_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1/7 & 1/7 & 0 \\ 1/49 & -1/49 & 1/7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 6.1224 & -5.1224 & 22.8571 \\ -0.6122 & 0.6122 & -3.2857 \\ 0.0204 & -0.0204 & 0.1429 \end{bmatrix}.$$

3 CONCLUSIONS

In several areas of applied mathematics and engineering (for instance in control and system theory), structural matrices appear regularly. The present paper has been motivated by a general, mathematical method of approximating distributional inputs with smooth functions that achieve a similar (control and system) objective. This fruitful idea and the relative mathematical techniques were first introduced by two significant applied engineers, Gupta and Hasdorff, see for more details [2]-[4] and [5]. Recently, the proposed method by Gupta and Hasdorff has been applied in [4] for the change of the initial state of an open loop, linear higher-order descriptor (regular) differential system in (almost) zero-time. This control and system theory application has created the need to calculate the generalized inverses of a structural matrice.

Consequently, in the present paper, two main results have been proposed: First, we have provided a (quasi) LU factorization, and secondly we have calculated analytically the generalized inverses of rectangular Special structure matrix, which is defined in terms of scalars $\mu, \lambda \in \mathbb{R}$ by the following expression:

$$\begin{bmatrix} 1 & \mu & \mu^2 & \mu^3 & * & \cdots & * & * & \cdots & \mu^{n-1} \\ 1 & \lambda & \lambda^2 & \lambda^3 & * & \cdots & * & * & \cdots & \lambda^{n-1} \\ 0 & 1 & 2\lambda & 3\lambda^2 & * & \cdots & * & * & \cdots & (n-1)\,\lambda^{n-2} \\ 0 & 0 & 1 & 3\lambda & * & \cdots & * & * & \cdots & (n-1)\,(n-2)\,\lambda^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & * & \cdots & \frac{1}{(m-1)}\frac{d^{m-1}}{d\lambda_j^{m-1}}\left(\lambda^{n-1}\right) \end{bmatrix} \in \mathcal{M}_{m+1,n}.$$

Finally, it should be mentioned that some other properties of the special structure matrix need to be studied further, and we leave them as a future research.

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