The 6th International Federation of Nonlinear Analysts Conference (2012) 77–87

Variational sequences on fibred velocity spaces

Zbyněk Urban[1] and Demeter Krupka[2]

[1]Department of Mathematics and Physics, University of Pardubice, Studentská 95, 532 10 Pardubice, Czech Republic, and Lepage Research Institute, Czech Republic, E-mail: urbanzp@gmail.com
[2]Lepage Research Institute, Czech Republic, and
Department of Mathematics, University of Ostrava, Czech Republic, and
Department of Mathematics, LaTrobe University, Victoria, Australia
E-mail: demeter.krupka@lepageri.eu
Webpage: http://www.lepageri.eu

ABSTRACT: The variational sequence theory in geometric mechanics is extended to second order velocity spaces over smooth manifolds. New explicit formulas for the classes in this sequence, representing the variational objects such as Lagrangians, Euler-Lagrange forms and Helmholtz forms, are derived. The expressions, given in the canonical coordinates, explain the structure of trivial Lagrangians on these underlying manifolds and allow straightforward applications in the inverse problem of the calculus of variations. The differences between local and global variationality are discussed and illustrated by examples. The variational theory of parameter-invariant problems of second order is considered in terms of jet differential groups.

KEYWORDS: variational sequence, contact form, Lagrangian, Euler-Lagrange expressions, Helmholtz conditions, invariant variational functional, Zermelo conditions, jet, fibred manifold

MSC (2010): 58E30, 58E99, 58A20, 49Q99, 49N45

1 INTRODUCTION

The variational sequence as considered in this paper was introduced by D. Krupka [2] as an adequate tool for the global characterization of the Euler-Lagrange mapping of the calculus of variations on *finite order* jet spaces. Elements of the variational sequence (classes of differential forms), represent the variational objects, well-known from the local variational theory. The crucial meaning of the sequence is that the sequence morphisms (variational mappings such as the Euler-Lagrange mapping, the Helmholtz mapping, etc.) can completely be determined without reference to the underlying variational functionals and, moreover, differencies between their local and global properties can exactly be formulated and characterized in terms of cohomology groups of underlying manifolds.

The goal in this paper is to study specific properties of variational sequences in higher order geometric mechanics when the underlying manifolds are velocity spaces. Then, having in mind applications, we derive basic coordinate variational formulas, esp. the Helmholtz variationality conditions, for 2nd order velocities.

Throughout, we use the Ehresmann's theory of jets and jet differential groups. A detailed exposition of the underlying structures we consider can be found in [5]; see also e.g. [1, 6, 9]. The basic notion of a differential form is used in definition of the variational integral; in this sense we follow the general variational theory on fibred manifolds, see [4] and the references therein. In Section 2, we give a review of the basic structures on underlying manifolds, including formulas for further computations. Section 3 briefly describes the key notion of the contact differential form. In Section 4, we present general properties of the variational sequence theory and then specify the results to second order differential forms on velocity manifolds. Beside local formulas, describing variational classes and mappings, we also give elementary examples characterizing local and global inverse problem of the calculus of variations on velocity spaces in terms of cohomology groups. Finally, in Section 5 we describe the variational functionals on second order velocity manifold, and study their parameter-invariance.

2 VELOCITIES AND GRASSMANN FIBRATIONS

Let Y be a smooth manifold of dimension m, $m \geq 1$. Let $r \geq 0$. We denote by T^rY the *manifold of velocities of order r over* Y. An element of T^rY, *a velocity of order r at a point y of* Y, is an r-jet $P \in J^r_{(0,y)}(\mathbb{R},Y)$, $P = J^r_0\zeta$, with source $0 \in \mathbb{R}$ and target $y = \zeta(0) \in Y$, represented by a curve ζ in Y. A velocity $P \in T^rY$ is said to be *regular*, if $P = J^r_0\zeta$ is represented by an *immersion* ζ at the origin $0 \in \mathbb{R}$. The set of regular velocities is denoted by $\mathrm{Imm}T^rY$. The canonical jet projections of T^rY onto T^sY, are denoted by $\tau^{r,s}$, where $0 \leq s \leq r$, i.e. $\tau^{r,s}(J^r_0\zeta) = J^s_0\zeta$. In the case of $r = 1$, T^1Y is the *tangent bundle* of Y, and velocities of order r are also called *tangent vectors* of order r

Let us recall the standard manifold structures of T^rY and $\mathrm{Imm}T^rY$. Every chart (V,ψ) on Y, where $\psi = (y^K) = (y^1,y^2,\dots,y^{m+1})$, induces a pair (V^r,ψ^r), with $V^r = (\tau^{r,0})^{-1}(V)$ and a collection of functions $\psi^r = (y^K,y^K_{(1)},y^K_{(2)},\dots,y^K_{(r)})$, defined on V^r by $y^K_{(l)}(J^r_0\zeta) = D^l(y^K\zeta)(0)$. Then the pairs (V^r,ψ^r) form charts on T^rY, called the *associated charts,* and define a smooth manifold structure on T^rY of dimension $(m+1)(r+1)$. As an open subset of T^rY, $\mathrm{Imm}T^rY$ is endowed with the structure of open submanifold. For the lower order case, we write $\dot{y}^K = y^K_{(1)}$, $\ddot{y}^K = y^K_{(2)}$, $\dddot{y}^K = y^K_{(3)}$.

Received August 15, 2012 IFNA Publishers, Inc.

The structure of $\mathrm{Imm}T^rY$ allows us to assign to every chart (V,ψ) on Y new collection of $(m+1)$ charts on $\mathrm{Imm}T^rY$, $(V^{r,L},\psi^{r,L})$, by shrinking the coordinate functions ψ^r to the domains

$$V^{r,L} = \{J_0^r\zeta \in V^r \,|\, \dot{y}^L(J_0^r\zeta) \neq 0\}.$$

For every index L, $1 \leq L \leq m+1$, we set $\psi^{r,L} = (y^L, y^L_{(1)}, y^L_{(2)}, \dots, y^L_{(r)}, y^\sigma, y^\sigma_{(1)}, y^\sigma_{(2)}, \dots, y^\sigma_{(r)})$, where the index σ runs through all values $1,2,\dots,m+1$, *not* equal to L. Clearly, the charts $(V^{r,L},\psi^{r,L})$ define a differentiable structure on $\mathrm{Imm}T^rY$.

The r-th order *differential group* L^r acts on manifold of velocities T^rY to the *right* by composition of jets,

$$T^rY \times L^r \ni (J_0^r\zeta, J_0^r\alpha) \to J_0^r\zeta \circ J_0^r\alpha = J_0^r(\zeta\circ\alpha) \in T^rY, \tag{1}$$

and clearly restricts to $\mathrm{Imm}T^rY$. Recall that an element of L^r is an r-jet $J_0^r\alpha$, represented by a diffeomorphism $\alpha: I \to J$ of open intervals in $\mathbb{R}$ such that $\alpha(0) = 0$. The group operation of L^r is defined by composition of jets, and L^r with this multiplication has a Lie group structure. The *canonical coordinates* on L^r are defined by $a_{(l)}(J_0^r\alpha) = D^l\alpha(0)$, $1 \leq l \leq r$; we denote $\dot{a} = a_{(1)}$, $\ddot{a} = a_{(2)}$, $\dddot{a} = a_{(3)}$.

We introduce on $\mathrm{Imm}T^rY$ another differentiable structure, related to action of the differential group L^r.

Lemma 1. *Let (V,ψ), $\psi = (y^K)$, be a chart on Y, and let $(V^{r,L},\psi^{r,L})$ be an associated chart on $\mathrm{Imm}T^rY$ for a fixed index L, $1 \leq L \leq m+1$. Then there exists a unique collection of functions $\chi^{r,L} = (w^L, w^L_{(1)}, w^L_{(2)}, \dots, w^L_{(r)}, w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r)$ on $V^{r,L}$ such that*

$$\begin{aligned} y^\sigma &= w^\sigma, \quad y^\sigma_{(l)} = \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} y^L_{|I_1|} y^L_{|I_2|} \dots y^L_{|I_p|} w^\sigma_p, \quad 1 \leq l \leq r, \\ y^L &= w^L, \quad y^L_{(1)} = w^L_{(1)}, \quad y^L_{(2)} = w^L_{(2)}, \quad \dots, \quad y^L_{(r)} = w^L_{(r)}, \end{aligned} \tag{2}$$

and $(V^{r,L},\chi^{r,L})$ form a chart on $\mathrm{Imm}T^rY$. Moreover, the functions $w^L, w^\sigma, w^\sigma_1, w^\sigma_2, \dots, w^\sigma_r$ are L^r-invariant.

The charts $(V^{r,L},\chi^{r,L})$, $1 \leq L \leq m+1$, define a differentiable structure on $\mathrm{Imm}T^rY$, and are referred to as *L-adapted* to the chart (V,ψ).

The following lemma describes the action (1) in canonical and adapted coordinates. Denote by $a_{(l)}, y^K_{(l)}, \bar{y}^K_{(l)}$ the canonical coordinates of $J_0^r\alpha$, $J_0^r\zeta$ and $J_0^r(\zeta\circ\alpha)$, and denote by $(w^L_{(l)}, w^\sigma_l)$ and $(\bar{w}^L_{(l)}, \bar{w}^\sigma_l)$ the adapted coordinates of $J_0^r\zeta$ and $J_0^r(\zeta\circ\alpha)$, respectively.

Lemma 2. *The group action* (1), *$(J_0^r\zeta, J_0^r\alpha) \to J_0^r\zeta \circ J_0^r\alpha$, is expressed in the associated chart $(V^{r,L},\psi^{r,L})$ by equations*

$$\bar{y}^K = y^K, \quad \bar{y}^K_{(l)} = \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} a_{|I_1|} a_{|I_2|} \dots a_{|I_p|} y^K_{(p)}, \quad 1 \leq l \leq r,$$

and, in the L-adapted chart $(V^{r,L},\chi^{r,L})$, by

$$\begin{aligned} \bar{w}^L &= w^L, \quad \bar{w}^L_{(l)} = \sum_{p=1}^{l} \sum_{(I_1,I_2,\dots,I_p)} a_{|I_1|} a_{|I_2|} \dots a_{|I_p|} w^L_{(p)}, \quad 1 \leq l \leq r, \\ \bar{w}^\sigma &= w^\sigma, \quad \bar{w}^\sigma_1 = w^\sigma_1, \quad \bar{w}^\sigma_2 = w^\sigma_2, \quad \dots, \quad \bar{w}^\sigma_r = w^\sigma_r. \end{aligned}$$

Let γ be a smooth curve in Y defined on an open interval $I \subset \mathbb{R}$. The curve $T^r\gamma$ in T^rY, defined by

$$I \ni t \to T^r\gamma(t) = J^r(\gamma \circ \mathrm{tr}_{-t}) \in T^rY, \tag{3}$$

is called the *r-jet prolongation of* γ.

In the following lemma we consider the basic properties of $T^r\gamma$.

Lemma 3. *Let (V,ψ) be a chart on Y. If $\gamma: I \to Y$ is a smooth curve, then the prolongation curve $T^r\gamma$* (3) *satisfies:*

(a) *The chart expression of $T^r\gamma$ in the associated chart (V^r,ψ^r) is given by*

$$y^K_{(l)} \circ T^r\gamma(t) = D(y^K_{(l-1)} \circ T^{r-1}\gamma)(t) = D^l(y^K \circ \gamma)(t),$$

and in the adapted chart $(V^{r,L},\chi^{r,L})$ by

$$w^\sigma_l \circ T^r\gamma(t) = D(w^\sigma_{l-1} \circ T^{r-1}\gamma \circ (w^L \circ \gamma)^{-1})(w^L(\gamma(t))), \;\; w^L_{(s)} \circ T^r\gamma(t) = D^s(w^L \circ \gamma)(t).$$

(b) For every diffeomorphism of open intervals $\mu: J \to I$,

$$T^r(\gamma \circ \mu)(s) = T^r\gamma(\mu(s)) \circ \mu^r(s), \tag{4}$$

where $\mu^r(s) = J_0^r(\mathrm{tr}_{\mu(s)} \circ \mu \circ \mathrm{tr}_{-s})$ belongs to L^r for all $s \in J$.

Although Y is *not* supposed to be a fibred manifold, we shall consider the associated trivially fibred manifold $\mathbb{R}\times Y$ over $\mathbb{R}$, with projection $\pi:\mathbb{R}\times Y\to\mathbb{R}$ and its product manifold structure. As it is usual, the *r-jet prolongation* of fibred manifold $\mathbb{R}\times Y$ is denoted by $J^r(\mathbb{R}\times Y)$, with elements r-jets $J^r_x\gamma$ of sections γ of π at a point $x\in\mathbb{R}$. The *canonical jet projections* are denoted by $\pi^r:J^r(\mathbb{R}\times Y)\to\mathbb{R}$ and $\pi^{r,s}:J^r(\mathbb{R}\times Y)\to J^s(\mathbb{R}\times Y)$; $\pi^r(J^r_x\gamma)=x$, $\pi^{r,s}(J^r_x\gamma)=J^s_x\gamma$. The *canonical r-jet prolongation* $J^r\gamma$ of a section γ of π is a curve in $J^r(\mathbb{R}\times Y)$, defined by $J^r\gamma(x)=J^r_x\gamma$.

The manifold of higher order velocities arise from the jet prolongation of a *fibred* manifold. We can *canonically identify* the r-jet prolongation $J^r(\mathbb{R}\times Y)$ of $\mathbb{R}\times Y$ with the product manifold $\mathbb{R}\times T^rY$ by means of the mapping $\phi^r:J^r(\mathbb{R}\times Y)\to\mathbb{R}\times T^rY$, defined by $\phi^r(J^r_x\gamma)=\left(x,J^r_0(\gamma_0\circ\mathrm{tr}_{-x})\right)$, where tr_{-x} is a translation $t\to t+x$ of $\mathbb{R}$, and γ is a section of $\mathbb{R}\times Y$, $\gamma(t)=(t,\gamma_0(t))$. The inverse of ϕ^r, $(\phi^r)^{-1}:\mathbb{R}\times T^rY\to J^r(\mathbb{R}\times Y)$, is then of the form $(\phi^r)^{-1}(x,J^r_0\zeta)=J^r_x\tilde{\zeta}$, where $\tilde{\zeta}(t)=(t,\zeta\circ\mathrm{tr}_x(t))$. Obviously, the mapping $\phi^r\circ J^r\gamma=\mathrm{id}_{\mathbb{R}}\times T^r\gamma_0$ is a section of trivially fibred manifold $\mathbb{R}\times T^rY$. Defining the canonical projection $\mathbb{R}\times T^rY\to\mathbb{R}\times T^sY$ by $\mathrm{id}_{\mathbb{R}}\times\tau^{r,s}$, we get the commutative diagram $(\mathrm{id}_{\mathbb{R}}\times\tau^{r,s})\circ\phi^r=\phi^s\circ\pi^{r,s}$; the induced canonical projection $\{0\}\times T^rY\to\{0\}\times T^sY$ we identify with $\tau^{r,s}$. For every open set $W\subset\mathbb{R}\times Y$, we denote $W^r=\phi^r((\pi^{r,0})^{-1}(W))\subset\mathbb{R}\times T^rY$, and $\mathrm{W}^{\mathrm{r}}_0=\mathrm{pr}_2(W^r)=(\tau^{r,0})^{-1}(\mathrm{pr}_2(W))\subset T^rY$, where pr_2 denotes the *second Cartesian projection* of $\mathbb{R}\times T^rY$.

We now introduce the concepts of formal derivative morphism and formal derivative of a function, adapted to our approach (cf. [8, 5]). Let an element $J^r_0\zeta\in T^rY$ be given. The representative ζ of $J^r_0\zeta$ induces the $(r-1)$-jet prolongation $T^{r-1}\zeta$, and the tangent mapping at the origin $0\in\mathbb{R}$, $T_0T^{r-1}\zeta$, sending a tangent vector of $T_0\mathbb{R}$ to a tangent vector of $T^{r-1}Y$ at the point $\tau^{r,r-1}(J^r_0\zeta)=J^{r-1}_0\zeta$. Denoting t the canonical coordinate on $\mathbb{R}$, we define the *vector field δ along the projection* $\tau^{r,r-1}$ by $\delta(J^r_0\zeta)=T_0T^{r-1}\zeta\cdot(d/dt)_0$, called the *formal derivative morphism of order r*. If (V,ψ), $\psi=(y^K)$, is a chart on Y, we get the coordinate expression of δ of the form

$$\delta(J^r_0\zeta)=\sum_{l=0}^{r-1}y^K_{(l+1)}(J^r_0\zeta)\left(\frac{\partial}{\partial y^K_{(l)}}\right)_{J^{r-1}_0\zeta}. \tag{5}$$

Let (V,ψ), $\psi=(y^K)$, be a chart on Y, and $f:V^{r-1}\to\mathbb{R}$ be a function. Formula (5) then induces a function $\delta(f):V^r\to\mathbb{R}$, called the *formal derivative* of function f. Restricting the formal derivative morphism δ to $\mathrm{Imm}T^rY$, we define an associated morphism $d/dw^L:\mathrm{Imm}T^rY\supset V^{r,L}\to T\mathrm{Imm}T^{r-1}Y$ by $d/dw^L=(1/\dot{y}^L)\delta$. It is not difficult to express d/dw^L in terms of an L-adapted chart $(V^{r,L},\chi^{r,L})$; we get

$$\frac{d}{dw^L}=\frac{\partial}{\partial w^L}+\sum_{l=0}^{r-1}w^\sigma_{l+1}\frac{\partial}{\partial w^\sigma_l}+\sum_{s=1}^{r-1}\frac{w^L_{(s+1)}}{\dot{w}^L}\frac{\partial}{\partial w^L_{(s)}}.$$

With respect to L-adapted chart $(V^{r,L},\chi^{r,L})$, the induced formal derivative of a function $f:V^{r,L}\to\mathbb{R}$ is denoted by df/dw^L.

The *formal derivative* of a function f, defined on an open subset of $J^{r-1}(\mathbb{R}\times Y)$, is in a fibred chart (U,φ), $\varphi=(t,y^K)$, on $\mathbb{R}\times Y$, denoted by df/dt.

Remark 1. (*Second order formulas*) The transformation formulas between second order charts $(V^{2,L},\psi^{2,L})$ and $(V^{2,L},\chi^{2,L})$ are given by

$$\begin{aligned} y^L &= w^L,\ \dot{y}^L=\dot{w}^L,\ \ddot{y}^L=\ddot{w}^L,\ y^\sigma=w^\sigma,\ \dot{y}^\sigma=w^\sigma_1\dot{w}^L,\ \ddot{y}^\sigma=w^\sigma_2(\dot{w}^L)^2+w^\sigma_1\ddot{w}^L,\\ w^L &= y^L,\ \dot{w}^L=\dot{y}^L,\ \ddot{w}^L=\ddot{y}^L,\ w^\sigma=y^\sigma,\ w^\sigma_1=\frac{\dot{y}^\sigma}{\dot{y}^L},\ w^\sigma_2=\frac{1}{(\dot{y}^L)^2}\left(\ddot{y}^\sigma-\frac{\ddot{y}^L}{\dot{y}^L}\dot{y}^\sigma\right),\end{aligned}$$

and the canonical group action (1) is expressed in associated and adapted charts by

$$\bar{y}^K=y^K,\ \dot{\bar{y}}^K=\dot{y}^K\dot{a},\ \ddot{\bar{y}}^K=\ddot{y}^K\dot{a}^2+\dot{y}^K\ddot{a},$$

$$\bar{w}^L=w^L,\ \dot{\bar{w}}^L=\dot{w}^L\dot{a},\ \ddot{\bar{w}}^L=\ddot{w}^L\dot{a}^2+\dot{w}^L\ddot{a},\ \bar{w}^\sigma=w^\sigma,\ \bar{w}^\sigma_1=w^\sigma_1,\ \bar{w}^\sigma_2=w^\sigma_2.$$

The chart expressions of the 2-jet prolongation $T^2\gamma$ of a curve γ in Y, are given by

$$\begin{aligned} w^L\circ T^2\gamma(t) &= w^L\circ\gamma(t),\ \dot{w}^L\circ T^2\gamma(t)=D(w^L\circ\gamma)(t),\ \ddot{w}^L\circ T^2\gamma(t)=D^2(w^L\circ\gamma)(t),\\ w^\sigma\circ T^2\gamma(t) &= w^\sigma\circ\gamma(t),\ w^\sigma_1\circ T^2\gamma(t)=D(w^\sigma\circ\gamma\circ(w^L\circ\gamma)^{-1})(w^L(\gamma(t))),\\ w^\sigma_2\circ T^2\gamma(t) &= D(w^\sigma_1\circ\gamma\circ(w^L\circ\gamma)^{-1})(w^L(\gamma(t))),\end{aligned}$$

and if $\mu:J\to I$ is a diffeomorphism of open intervals, we get from Lemma 3, (b), expressions for the curve $T^2(\gamma\circ\mu)$ on J,

$$\begin{aligned} w^L\circ T^2(\gamma\circ\mu)(s) &= w^L\circ T^2\gamma(\mu(s)),\ \dot{w}^L\circ T^2(\gamma\circ\mu)(s)=\dot{w}^L\circ T^2\gamma(\mu(s))\dot{a}(\mu^2(s)),\\ \ddot{w}^L\circ T^2(\gamma\circ\mu)(s) &= \ddot{w}^L\circ T^2\gamma(\mu(s))\dot{a}(\mu^2(s))^2+\dot{w}^L\circ T^2\gamma(\mu(s))\ddot{a}(\mu^2(s)),\\ w^\sigma\circ T^2(\gamma\circ\mu)(s) &= w^\sigma\circ T^2\gamma(\mu(s)),\ w^\sigma_1\circ T^2(\gamma\circ\mu)(s)=w^\sigma_1\circ T^2\gamma(\mu(s)),\\ w^\sigma_2\circ T^2(\gamma\circ\mu)(s) &= w^\sigma_2\circ T^2\gamma(\mu(s)),\end{aligned}$$

where $\mu^2(s)=J^2_0(\mathrm{tr}_{\mu(s)}\circ\mu\circ\mathrm{tr}_{-s})$ is an element of second order differential group L^2.

3 DIFFERENTIAL FORMS

Let W be an open subset of manifold $\mathbb{R}\times Y$, and W^r the r-jet prolongation of W in $\mathbb{R}\times T^rY$. Let $\Omega_0^r W$ denotes the ring of smooth functions defined on W^r, and $\Omega_k^r W$ the $\Omega_0^r W$-module of smooth differential k-forms defined on W^r.

We say that a 1-form $\rho\in\Omega_1^r W$ is *contact*, if

$$(\mathrm{id}_{\mathbb{R}}\times T^r\zeta)^*\rho=0 \tag{6}$$

for all curves ζ in Y, defined on the open set $\pi(W)\subset\mathbb{R}$.

In next sections we shall consider the forms $\eta\in\Omega_k^r W$, defined on $\mathrm{W}_0^{\mathrm{r}}=\mathrm{pr}_2(W^r)\subset T^rY$, where pr_2 denotes the *second Cartesian projection* of $\mathbb{R}\times T^rY$. These forms define $\Omega_{0,0}^r W$-module, denoted by $\Omega_{k,0}^r W$, where $\Omega_{0,0}^r W$ is a ring of smooth functions on W_0^r.

The definition of contactness (6) now reduces for differential forms on velocity spaces in the following sense. We say that a 1-form $\rho\in\Omega_{1,0}^r W$ is *contact*, if

$$(T^r\zeta)^*\rho=0 \tag{7}$$

for all smooth curves ζ in Y, defined on the open set $\pi(W)\subset\mathbb{R}$.

Note that both of these definitions of contactness (6) and (7) imply that every function f is contact if and only if f vanishes identically, and that every k-form is contact for $k\geq 2$.

The following lemma describes the local structure of contact 1-forms of $\Omega_1^r W$ and $\Omega_{1,0}^r W$.

Lemma 4. (a) *Let (U,φ), $\varphi=(t,y^K)$, be a chart on $\mathbb{R}\times Y$ such that $U\subset W$. Let $\rho\in\Omega_1^r W$ be a 1-form, locally expressed by $\rho=Adt+\sum_{l=0}^r B_K^l dy_{(l)}^K$. Then ρ is contact if and only if*

$$\rho=\sum_{l=0}^{r-1} B_K^l\omega_{(l)}^K,$$

where

$$\omega_{(l)}^K=dy_{(l)}^K-y_{(l+1)}^K dt,\quad 0\leq l\leq r-1. \tag{8}$$

(b) *Let (V,ψ), $\psi=(y^K)$, be a chart on Y such that $\pi(W)\times V\subset W$. Let $\rho\in\Omega_{1,0}^r W$ be a 1-form, locally expressed by $\rho=\sum_{l=0}^r A_K^l dy_{(l)}^K$. Then ρ is contact if and only if*

$$A_K^r=0,\quad \sum_{l=0}^{r-1} A_K^l y_{(l+1)}^K=0.$$

Moreover, if ρ is expressed in an L-adapted chart $(V^{r,L},\psi^{r,L})$ on $\mathrm{W}_0^{\mathrm{r}}\cap\mathrm{Imm}T^rY$ by $\rho=\sum_{l=0}^r A_\sigma^l dy_{(l)}^\sigma+\sum_{s=0}^r A_L^s dy_{(s)}^L$ (no summation through L), then ρ is contact if and only if

$$\rho=\sum_{l=0}^{r-1} A_\sigma^l\eta_{(l)}^\sigma+\sum_{s=1}^{r-1} A_L^s\eta_{(s)}^L,$$

where

$$\eta_{(l)}^\sigma=dy_{(l)}^\sigma-\frac{y_{(l+1)}^\sigma}{\dot{y}^L}dy^L,\quad \eta_{(s)}^L=dy_{(s)}^L-\frac{y_{(s+1)}^L}{\dot{y}^L}dy^L, \tag{9}$$

and if $\rho=\sum_{l=0}^r A_\sigma^l dw_l^\sigma+\sum_{s=0}^r A_L^s dw_{(s)}^L$ (no summation through L) in an L-adapted chart $(V^{r,L},\chi^{r,L})$ on $\mathrm{W}_0^{\mathrm{r}}\cap\mathrm{Imm}T^rY$, then ρ is contact if and only if

$$\rho=\sum_{l=0}^{r-1} A_\sigma^l\eta_l^\sigma+\sum_{s=1}^{r-1} A_L^s\eta_{(s)}^L,$$

where

$$\eta_l^\sigma=dw_l^\sigma-w_{l+1}^\sigma dw^L,\quad \eta_{(s)}^L=dw_{(s)}^L-\frac{w_{(s+1)}^L}{\dot{w}^L}dw^L. \tag{10}$$

Proof. Both assertions (a) and (b) immediately follow from a calculation of pull-back in the definition of contactness (6) and (7), respectively.

Remark 2. We note that if a 1-form $\rho\in\Omega_{1,0}^r W$ is contact in sense of (10) (or (9)), then it is also contact in sense of (8), after the lift by means of second canonical projection $\mathrm{pr}_2:\mathbb{R}\times\mathrm{Imm}T^rY\to\mathrm{Imm}T^rY$. In particular, the contact forms η_l^σ, $\eta_{(s)}^L$ (10) of $\Omega_{1,0}^r W$ transform to contact forms $\omega_{(l)}^K$ (8) of $\Omega_1^r W$. For example, if $r=2$, we get

$$\eta^\sigma=\omega^\sigma-\frac{\dot{y}^\sigma}{\dot{y}^L}\omega^L,\ \ \eta_1^\sigma=\frac{1}{\dot{y}^L}\dot{\omega}^\sigma-\frac{1}{(\dot{y}^L)^2}\left(\ddot{y}^\sigma-\frac{\ddot{y}^L}{\dot{y}^L}\dot{y}^\sigma\right)\omega^L-\frac{\dot{y}^\sigma}{(\dot{y}^L)^2}\dot{\omega}^L,\ \ \dot{\eta}^L=\dot{\omega}^L-\frac{\ddot{y}^L}{\dot{y}^L}\omega^L.$$

We note that, clearly, 1-forms (8), $\omega^K_{(l)}$, $0 \le l \le r-1$, and (10), η^σ_l, $\eta^L_{(s)}$, where $0 \le l \le r-1$, $1 \le s \le r-1$, are contact and linearly independent. The 1-forms $\{dt,\ \omega^K_{(l)},\ dy^K_{(r)}\}$, $\{dw^L,\ \eta^\sigma_l,\ \eta^L_{(s)},\ dw^\sigma_r,\ dw^L_r\}$, define a *contact basis* of linear forms on W^r and W^r_0, respectively. As usual, for $r \le 3$ we denote the contact 1-forms (8) by $\omega^K = \omega^K_{(0)}$, $\dot{\omega}^K = \omega^K_{(1)}$, $\ddot{\omega}^K = \omega^K_{(2)}$, and $\dddot{\omega}^K = \omega^K_{(3)}$.

The ideal of the exterior algebra of differential forms, generated by contact 1-forms, is called the *contact ideal.* A differential form is said to be *contact*, if it belongs to the contact ideal. A contact form, containing exactly k exterior factors (8), resp. (10), is said to be *k-contact*.

It is well-known that k-contact forms, generated by contact 1-forms (8), form a submodule of the module $\Omega^r_k W$ of differential forms, defined on a prolongation of a fibred manifold; in our case $J^r(\mathbb{R}\times Y)$ isomorphic to $\mathbb{R}\times T^r Y$.

Lemma 5. *Let W be an open subset of $\mathbb{R}\times Y$, and (V_1,ψ_1), $\psi_1 = (y^K)$, (V_2,ψ_2), $\psi_2 = (\bar{y}^K)$, be two charts on Y such that $V_1, V_2 \subset \mathrm{pr}_2(W)$, $V_1 \cap V_2 \neq \emptyset$. If $(V_1^{r,L},\chi_1^{r,L})$ and $(V_2^{r,M},\chi_2^{r,M})$ are two adapted charts on $W^r_0 \cap \mathrm{Imm}T^rY$, then*

$$\begin{aligned}
\bar{\eta}^\nu_p &= \left(\frac{\partial \bar{w}^\nu_p}{\partial w^\sigma} + \bar{w}^\nu_{p+1}\frac{\partial \bar{w}^M}{\partial w^\sigma}\right)\eta^\sigma + \sum_{l=1}^{p}\frac{\partial \bar{w}^\nu_p}{\partial w^\sigma_l}\eta^\sigma_l + \sum_{s=1}^{p}\frac{\partial \bar{w}^\nu_p}{\partial w^L_{(s)}}\eta^L_{(s)},\\
\bar{\eta}^M_{(q)} &= \left(\frac{\partial \bar{w}^M_{(q)}}{\partial w^\sigma} + \frac{\bar{w}^M_{(q+1)}}{\dot{\bar{w}}^M}\frac{\partial \bar{w}^M}{\partial w^\sigma}\right)\eta^\sigma + \sum_{l=1}^{q}\frac{\partial \bar{w}^M_{(q)}}{\partial w^\sigma_l}\eta^\sigma_l + \sum_{s=1}^{q}\frac{\partial \bar{w}^M_{(q)}}{\partial w^L_{(s)}}\eta^L_{(s)},
\end{aligned}$$

where $0 \le p \le r-1$, and $1 \le q \le r-1$.

Proof. The transformation properties are obtained by a straightforward calculation.

Corollary. *k-contact forms on $W^r_0 \cap \mathrm{Imm}T^rY$, constitute a submodule of the module of differential forms $\Omega^r_{k,0}W$.*

It is the standard result that the pull-back of a differential form $\rho \in \Omega^r_k W$, by means of the canonical jet projection $\pi^{r+1,r}$, can be uniquely decomposed into its contact components (see e.g. [3]). We have

$$(\pi^{r+1,r})^*\rho = p_{k-1}\rho + p_k\rho, \tag{11}$$

where $p_{k-1}\rho$ is the $(k-1)$-*contact component*, and $p_k\rho$ is the *k-contact component* of ρ. We note that an analogous contact decomposition formula holds also for differential forms of $\Omega^r_{k,0}W$, defined on an open subset of manifold of regular velocities $\mathrm{Imm}T^rY$ (see [9]).

4 THE VARIATIONAL SEQUENCE

Let Y be a smooth manifold, $\dim Y = m$. Let $W \subset \mathbb{R}\times Y$ be an open set, W^r its r-jet prolongation to $\mathbb{R}\times T^rY$, and $\Omega^r_k W$ be the module of smooth differential k-forms, defined on W^r. Let ${}^{(c)}\Omega^r_k W$ denotes the submodule of $\Omega^r_k W$ of k-contact k-forms. Denote

$${}^{(c)}\Omega^r_0 W = \{0\},\ \ \Theta^r_k W = {}^{(c)}\Omega^r_k W + d\,{}^{(c)}\Omega^r_{k-1}W, \tag{12}$$

in the sense that a k-form $\rho \in \Omega^r_k W$ belongs to $\Theta^r_k W$ if and only if ρ has a local decomposition $\rho = \rho_0 + d\rho'$ for some $\rho_0 \in {}^{(c)}\Omega^r_k W$ and $\rho' \in {}^{(c)}\Omega^r_{k-1}W$. $\Theta^r_k W$ is a subgroup of the Abelian group $\Omega^r_k W$, and we get a subsequence of Abelian groups

$$0 \to \Theta^r_1 W \to \Theta^r_2 W \to \ldots \to \Theta^r_M W \to 0 \tag{13}$$

of the DeRham sequence

$$0 \to \mathbb{R} \to \Omega^r_0 W \to \Omega^r_1 W \to \ldots \to \Omega^r_M W \to \Omega^r_{M+1} W \to \ldots \to \Omega^r_N W \to 0, \tag{14}$$

where the morphisms denote the *exterior derivative* d, and $M = mr+1$, $N = \dim(\mathbb{R}\times T^rY) = m(r+1)+1$.

Theorem 1. *Let W be an open set in $\mathbb{R}\times Y$. Then $\Theta^r_k W$ is a direct sum of the module ${}^{(c)}\Omega^r_k W$ and of the image of the module ${}^{(c)}\Omega^r_{k-1}W$ in exterior derivative operator d, i.e. if $\rho \in \Theta^r_k W$, then there exist unique forms $\rho_0 \in {}^{(c)}\Omega^r_k W$ and $\rho' \in {}^{(c)}\Omega^r_{k-1}W$ such that $\rho = \rho_0 + d\rho'$ on a neighbourhood in W^r.*

Proof. It is sufficient to show uniqueness of the decomposition $\rho = \rho_0 + d\rho'$ in some chart on W^r. The proof is based on the structure of k-contact and $(k-1)$-contact forms, and in case of arbitrary fibred manifold with one-dimensional base it can be found in Krupka [3].

As a consequence of Theorem 1 and *Poincaré lemma* for contact forms (cf. [3]), we observe that the subsequence (13) is *exact*, and it is said to be the *contact subsequence* of the DeRham sequence (14).

The quotient sequence

$$0 \to \mathbb{R} \to \Omega^r_0 W \to \Omega^r_1 W/\Theta^r_1 W \to \ldots \to \Omega^r_M W/\Theta^r_M W \to \Omega^r_{M+1} W \to \ldots \to \Omega^r_N W \to 0, \tag{15}$$

where the quotient mappings $E : \Omega^r_k W/\Theta^r_k W \to \Omega^r_{k+1} W/\Theta^r_{k+1} W$ are defined by

$$E([\rho]) = [d\rho], \tag{16}$$

is then also exact. In (16), $[\rho]$ denotes the class of a form ρ. It is easily seen that the mappings (16) are well-defined. The sequence (15) is said to be the *variational sequence of order r on* $\mathbb{R}\times T^rY$ (cf. [3, 8]).

For concrete coordinate calculations of classes, entering the variational sequence, the following result is important. For any form $\rho\in\Omega^r_kW$, the class of ρ and the class of lifted form $(\tau^{r+1,r})^*\rho$ can be identified.

Theorem 2. *The quotient mapping* $\Omega^r_kW/\Theta^r_kW\to\Omega^{r+1}_kW/\Theta^{r+1}_kW$ *is injective.*

Proof. We refer to [3] and [9].

Classes of forms on T^2Y

The following lemma describes, by means of canonical coordinates, the classes of differential forms on T^2Y, as elements of the *second order* variational sequence on fibred velocity manifold $\mathbb{R}\times T^2Y$.

Lemma 6. *Let* (U,φ), $\varphi=(t,y^K)$, *be a chart on* $\mathbb{R}\times Y$, *and* (U^2,φ^2), $\varphi^2=(t,y^K,\dot y^K,\ddot y^K)$, *the associated chart on* $W^2\subset\mathbb{R}\times T^2Y$.

(a) Let $\rho\in\Omega^2_{1,0}W$ *be locally expressed by* $\rho=A_Kdy^K+\dot A_Kd\dot y^K+\ddot A_Kd\ddot y^K$. *Then the class* $[\rho]$ *is an element of* Ω^3_1W/Θ^3_1W, *given by*

$$[\rho]=\left(A_K\dot y^K+\dot A_K\ddot y^K+\ddot A_K\dddot y^K\right)dt. \tag{17}$$

(b) *Let* $\rho\in\Omega^2_{2,0}W$ *be locally expressed by*

$$\begin{aligned}\rho &= \frac{1}{2}A_{KM}dy^K\wedge dy^M+\dot A_{K,M}d\dot y^K\wedge dy^M+\ddot A_{K,M}d\ddot y^K\wedge dy^M\\ &+ \frac{1}{2}B_{KM}d\dot y^K\wedge d\dot y^M+B_{K,M}d\ddot y^K\wedge d\dot y^M+\frac{1}{2}C_{KM}d\ddot y^K\wedge d\ddot y^M.\end{aligned} \tag{18}$$

Then the class $[\rho]$ *is an element of* Ω^5_2W/Θ^5_2W, *given by*

$$[\rho]=E_K([\rho])\omega^K\wedge dt, \tag{19}$$

where

$$\begin{aligned}E_K([\rho]) &= A_{KM}\dot y^M-\dot A_{M,K}\ddot y^M-\ddot A_{M,K}\dddot y^M-\frac{d}{dt}(\dot A_{K,M}\dot y^M+B_{KM}\ddot y^M-B_{M,K}\dddot y^M)\\ &+\frac{d^2}{dt^2}(\ddot A_{K,M}\dot y^M+B_{K,M}\ddot y^M+C_{KM}\dddot y^M).\end{aligned}$$

(c) *Let* $\rho\in\Omega^2_{3,0}W$ *be locally expressed by*

$$\begin{aligned}\rho &= \frac{1}{6}A_{KMN}dy^K\wedge dy^M\wedge dy^N+\frac{1}{2}\dot A_{K,MN}d\dot y^K\wedge dy^M\wedge dy^N+\frac{1}{2}\dot A_{KM,N}d\dot y^K\wedge d\dot y^M\wedge dy^N\\ &+ \frac{1}{2}\ddot A_{K,MN}d\ddot y^K\wedge dy^M\wedge dy^N+\ddot A_{K,M,N}d\ddot y^K\wedge d\dot y^M\wedge dy^N+\frac{1}{2}\ddot A_{KM,N}d\ddot y^K\wedge d\ddot y^M\wedge dy^N\\ &+ \frac{1}{6}B_{KMN}d\dot y^K\wedge d\dot y^M\wedge d\dot y^N+\frac{1}{2}B_{K,MN}d\ddot y^K\wedge d\dot y^M\wedge d\dot y^N+\frac{1}{2}B_{KM,N}d\ddot y^K\wedge d\ddot y^M\wedge d\dot y^N\\ &+ \frac{1}{6}C_{KMN}d\ddot y^K\wedge d\ddot y^M\wedge d\ddot y^N.\end{aligned}$$

Then the class $[\rho]$ *is an element of* Ω^7_3W/Θ^7_3W, *given by*

$$\begin{aligned}[\rho] &= \frac{1}{2}E^0_{MK}([\rho])\omega^M\wedge\omega^K\wedge dt+E^1_{M,K}([\rho])\dot\omega^M\wedge\omega^K\wedge dt+\frac{1}{2}E^2_{MK}([\rho])\ddot\omega^M\wedge\omega^K\wedge dt\\ &+ E^3_{M,K}([\rho])\dddot\omega^M\wedge\omega^K\wedge dt+\frac{1}{2}E^4_{MK}([\rho])\omega^M_{(4)}\wedge\omega^K\wedge dt,\end{aligned} \tag{20}$$

where

$$\begin{aligned}E^4_{MK}([\rho]) &= \ddot A_{MK,N}\dot y^N+B_{MK,N}\ddot y^N+C_{MKN}\dddot y^N,\\ E^3_{M,K}([\rho]) &= -\frac{1}{2}\left((\ddot A_{M,K,N}+\ddot A_{K,M,N})\dot y^N+(B_{M,KN}+B_{K,MN})\ddot y^N+(B_{NM,K}+B_{NK,M})\dddot y^N\right),\\ E^2_{MK}([\rho]) &= (\dot A_{KM,N}-\ddot A_{K,MN}+\ddot A_{M,KN})\dot y^N+(B_{KMN}+\ddot A_{K,N,M}-\ddot A_{M,N,K})\ddot y^N\\ &\quad+(B_{N,KM}-\ddot A_{NK,M}+\ddot A_{NM,K})\dddot y^N\\ &\quad-\frac{1}{2}\frac{d}{dt}\left((\ddot A_{K,M,N}-\ddot A_{M,K,N})\dot y^N+(B_{K,MN}-B_{M,KN})\ddot y^N+(B_{NK,M}-B_{NM,K})\dddot y^N\right)\\ &\quad+2\frac{d^2}{dt^2}\left(\ddot A_{KM,N}\dot y^N+B_{KM,N}\ddot y^N+C_{KMN}\dddot y^N\right),\end{aligned}$$

$$
\begin{aligned}
E^1_{M,K}([\rho]) &= \frac{1}{2}\Big((\dot{A}_{M,KN}+\dot{A}_{K,MN})\dot{y}^N + (\dot{A}_{NM,K}+\dot{A}_{NK,M})\ddot{y}^N + (\ddot{A}_{N,M,K}+\ddot{A}_{N,K,M})\dddot{y}^N \\
&\quad -\frac{d}{dt}\left((\ddot{A}_{M,KN}+\ddot{A}_{K,MN})\dot{y}^N - (\ddot{A}_{M,N,K}+\ddot{A}_{K,N,M})\ddot{y}^N + (\ddot{A}_{NM,K}+\ddot{A}_{NK,M})\dddot{y}^N \right) \\
&\quad +\frac{d^2}{dt^2}\left((\ddot{A}_{M,K,N}+\ddot{A}_{K,M,N})\dot{y}^N + (B_{M,KN}+B_{K,MN})\ddot{y}^N + (B_{NM,K}+B_{NK,M})\dddot{y}^N \right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
E^0_{MK}([\rho]) &= A_{MKN}\dot{y}^N - \dot{A}_{N,KM}\ddot{y}^N - \ddot{A}_{N,KM}\dddot{y}^N \\
&\quad -\frac{1}{2}\frac{d}{dt}\left((\dot{A}_{M,KN}-\dot{A}_{K,MN})\dot{y}^N + (\dot{A}_{NM,K}-\dot{A}_{NK,M})\ddot{y}^N + (\ddot{A}_{N,M,K}-\ddot{A}_{N,K,M})\dddot{y}^N \right) \\
&\quad +\frac{1}{2}\frac{d^2}{dt^2}\left(\dot{A}_{MK,N}\dot{y}^N + B_{MKN}\ddot{y}^N + B_{N,MK}\dddot{y}^N \right) \\
&\quad -\frac{1}{4}\frac{d^3}{dt^3}\left((\ddot{A}_{M,K,N}-\ddot{A}_{K,M,N})\dot{y}^N + (B_{M,KN}-B_{K,MN})\ddot{y}^N + (B_{NM,K}-B_{NK,M})\dddot{y}^N \right) \\
&\quad +\frac{1}{2}\frac{d^4}{dt^4}\left(\ddot{A}_{MK,N}\dot{y}^N + B_{MK,N}\ddot{y}^N + C_{MKN}\dddot{y}^N \right).
\end{aligned}
$$

Proof. We compute the pull-backs of η in canonical projections $\tau^{r,s}:\mathbb{R}\times T^rY\to\mathbb{R}\times T^sY$, and apply the contact decomposition of forms (11), together with the property $d\omega^K_{(l)} = -\omega^K_{(l+1)}\wedge dt$. The lifted form η is then factorized by means of contact forms (12).

Now, we find the local structure of the quotient mappings $E:\Omega^r_0W\to\Omega^r_1W/\Theta^r_1W$, $E:\Omega^r_1W/\Theta^r_1W\to\Omega^r_2W/\Theta^r_2W$, and $E:\Omega^r_2W/\Theta^r_2W\to\Omega^r_3W/\Theta^r_3W$, which appear in the sequence (15).

Lemma 7. *Let (U,φ), $\varphi=(t,y^K)$, be a chart on $\mathbb{R}\times Y$, and (U^2,φ^2), $\varphi^2=(t,y^K,\dot{y}^K,\ddot{y}^K)$, the associated chart on $W^2\subset\mathbb{R}\times T^2Y$.*

(a) *For a function $f\in\Omega^2_{0,0}W$ it holds $E(f)=E([f])=(df/dt)dt$.*

(b) *If 1-form $\rho\in\Omega^2_{1,0}W$ is locally expressed by $\rho=A_Kdy^K+\dot{A}_Kd\dot{y}^K+\ddot{A}_Kd\ddot{y}^K$, then*

$$E([\rho]) = E_K([d\rho])\omega^K\wedge dt, \tag{21}$$

where

$$
\begin{aligned}
E_K([d\rho]) &= \frac{\partial A_M}{\partial y^K}\dot{y}^M + \frac{\partial \dot{A}_M}{\partial y^K}\ddot{y}^M + \frac{\partial \ddot{A}_M}{\partial y^K}\dddot{y}^M - \frac{d}{dt}\left(\frac{\partial A_M}{\partial \dot{y}^K}\dot{y}^M + A_K + \frac{\partial \dot{A}_M}{\partial \dot{y}^K}\ddot{y}^M + \frac{\partial \ddot{A}_M}{\partial \dot{y}^K}\dddot{y}^M\right) \\
&\quad +\frac{d^2}{dt^2}\left(\frac{\partial A_M}{\partial \ddot{y}^K}\dot{y}^M + \frac{\partial \dot{A}_M}{\partial \ddot{y}^K}\ddot{y}^M + \dot{A}_K + \frac{\partial \ddot{A}_M}{\partial \ddot{y}^K}\dddot{y}^M\right) - \frac{d^3\ddot{A}_K}{dt^3}.
\end{aligned}
$$

(c) *If 2-form $\rho\in\Omega^2_{2,0}W$ is locally expressed by (26), then*

$$
\begin{aligned}
E([\rho]) &= \frac{1}{2}E^0_{MK}([d\rho])\omega^M\wedge\omega^K\wedge dt + E^1_{M,K}([d\rho])\dot{\omega}^M\wedge\omega^K\wedge dt \\
&\quad +\frac{1}{2}E^2_{MK}([d\rho])\ddot{\omega}^M\wedge\omega^K\wedge dt + E^3_{M,K}([d\rho])\dddot{\omega}^M\wedge\omega^K\wedge dt \\
&\quad +\frac{1}{2}E^4_{MK}([d\rho])\omega^M_{(4)}\wedge\omega^K\wedge dt,
\end{aligned}
\tag{22}
$$

where the coefficients are determined by a class (20) *of* 3*-form $d\rho$.*

Proof. By the definition of morphisms E (16), we apply formulas for classes from Lemma 6 to exterior derivative of forms.

The classes of exterior derivatives of forms, determined by Lemma 7, are closely related to variational objects, well-known from local variational theory.

Let (U,φ), $\varphi=(t,y^K)$, be a chart on $\mathbb{R}\times Y$, and consider a 1-form $\rho\in\Omega^2_{1,0}W$, locally expressed by

$$\rho = A_Kdy^K + \dot{A}_Kd\dot{y}^K + \ddot{A}_Kd\ddot{y}^K. \tag{23}$$

We define a *Lagrange function* $\mathscr{L}:V^3\to\mathbb{R}$ by

$$\mathscr{L} = A_K\dot{y}^K + \dot{A}_K\ddot{y}^K + \ddot{A}_K\dddot{y}^K, \tag{24}$$

and the corresponding *Euler-Lagrange expressions* $\varepsilon_K(\mathscr{L}):V^5\to\mathbb{R}$ by

$$\varepsilon_K(\mathscr{L}) = \frac{\partial\mathscr{L}}{\partial y^K} - \frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{y}^K} + \frac{d^2}{dt^2}\frac{\partial\mathscr{L}}{\partial\ddot{y}^K} - \frac{d^3}{dt^3}\frac{\partial\mathscr{L}}{\partial\dddot{y}^K}. \tag{25}$$

Note that by Lemma 6, we have $[\rho] = \mathscr{L} dt$.

Consider now a 2-form $\rho \in \Omega^2_{2,0} W$, locally expressed by (18). Denote

$$\begin{aligned} \varepsilon_P &= A_{PM}\dot{y}^M - \dot{A}_{M,P}\ddot{y}^M - \ddot{A}_{M,P}\dddot{y}^M - \frac{d}{dt}(\dot{A}_{P,M}\dot{y}^M + B_{PM}\ddot{y}^M - B_{M,P}\dddot{y}^M) \\ &+ \frac{d^2}{dt^2}(\ddot{A}_{P,M}\dot{y}^M + B_{P,M}\ddot{y}^M + C_{PM}\dddot{y}^M). \end{aligned} \tag{26}$$

Then by Lemma 6, $[\rho] = \varepsilon_P \omega^P \wedge dt$, and we define the corresponding *Helmholtz expressions* by

$$\begin{aligned} \mathscr{H}^5_{KM}(\varepsilon_P) &= \frac{\partial \varepsilon_K}{\partial y^M_{(5)}} + \frac{\partial \varepsilon_M}{\partial y^K_{(5)}}, \\ \mathscr{H}^4_{KM}(\varepsilon_P) &= \frac{\partial \varepsilon_K}{\partial y^M_{(4)}} - \frac{\partial \varepsilon_M}{\partial y^K_{(4)}} - \frac{5}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_K}{\partial y^M_{(5)}} - \frac{\partial \varepsilon_M}{\partial y^K_{(5)}}\right), \\ \mathscr{H}^3_{KM}(\varepsilon_P) &= \frac{1}{2}\left(\frac{\partial \varepsilon_K}{\partial y^M_{(3)}} + \frac{\partial \varepsilon_M}{\partial y^K_{(3)}} - 2\frac{d}{dt}\left(\frac{\partial \varepsilon_K}{\partial y^M_{(4)}} + \frac{\partial \varepsilon_M}{\partial y^K_{(4)}}\right)\right), \\ \mathscr{H}^2_{KM}(\varepsilon_P) &= \frac{\partial \varepsilon_K}{\partial y^M_{(2)}} - \frac{\partial \varepsilon_M}{\partial y^K_{(2)}} - \frac{3}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_K}{\partial y^M_{(3)}} - \frac{\partial \varepsilon_M}{\partial y^K_{(3)}}\right) + \frac{5}{2}\frac{d^3}{dt^3}\left(\frac{\partial \varepsilon_K}{\partial y^M_{(5)}} - \frac{\partial \varepsilon_M}{\partial y^K_{(5)}}\right), \\ \mathscr{H}^1_{KM}(\varepsilon_P) &= \frac{1}{2}\left(\frac{\partial \varepsilon_K}{\partial y^M_{(1)}} + \frac{\partial \varepsilon_M}{\partial y^K_{(1)}} - \frac{d}{dt}\left(\frac{\partial \varepsilon_K}{\partial y^M_{(2)}} + \frac{\partial \varepsilon_M}{\partial y^K_{(2)}}\right) + \frac{d^3}{dt^3}\left(\frac{\partial \varepsilon_K}{\partial y^M_{(4)}} - \frac{\partial \varepsilon_M}{\partial y^K_{(4)}}\right)\right), \\ \mathscr{H}^0_{KM}(\varepsilon_P) &= \frac{\partial \varepsilon_K}{\partial y^M} - \frac{\partial \varepsilon_M}{\partial y^K} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_K}{\partial y^M_{(1)}} - \frac{\partial \varepsilon_M}{\partial y^K_{(1)}}\right) + \frac{1}{4}\frac{d^3}{dt^3}\left(\frac{\partial \varepsilon_K}{\partial y^M_{(3)}} - \frac{\partial \varepsilon_M}{\partial y^K_{(3)}}\right) \\ &- \frac{1}{2}\frac{d^5}{dt^5}\left(\frac{\partial \varepsilon_K}{\partial y^M_{(5)}} - \frac{\partial \varepsilon_M}{\partial y^K_{(5)}}\right). \end{aligned} \tag{27}$$

From Lemma 7, (a), we see that the morphism $E : \Omega^r_0 W \to \Omega^r_1 W / \Theta^r_1 W$ in the first column of the variational sequence (15) is characterized by the *total derivative operator*. The following theorem characterizes the morphisms of (15) in the second and third columns; the mappings $E : \Omega^r_1 W / \Theta^r_1 W \to \Omega^r_2 W / \Theta^r_2 W$ and $E : \Omega^r_2 W / \Theta^r_2 W \to \Omega^r_3 W / \Theta^r_3 W$.

,

Theorem 3. *Let* (U, φ), $\varphi = (t, y^K)$, *be a chart on* $\mathbb{R} \times Y$, *and* (U^2, φ^2), $\varphi^2 = (t, y^K, \dot{y}^K, \ddot{y}^K)$, *the associated chart on* $W^2 \subset \mathbb{R} \times T^2 Y$.

(a) *If a* 1*-form* $\rho \in \Omega^2_{1,0} W$ *is locally expressed by* (23)*, then the coeficients of* $[d\rho]$ (21) *coincide with the Euler-Lagrange expressions of the associated Lagrange function* $\mathscr{L}$ (24),

$$E_K([d\rho]) = \varepsilon_K(\mathscr{L}).$$

(b) *If a* 2*-form* $\rho \in \Omega^2_{2,0} W$ *is locally expressed by* (18), *then the coeficients of* $[d\rho]$ (22) *coincide with the Helmholtz expressions of* ε_P (26)*,*

$$\begin{aligned} E^0_{MK}([d\rho]) &= \mathscr{H}^0_{KM}(\varepsilon_P), \quad E^1_{M,K}([d\rho]) = \mathscr{H}^1_{KM}(\varepsilon_P), \quad E^2_{MK}([d\rho]) = \mathscr{H}^2_{KM}(\varepsilon_P), \\ E^3_{M,K}([d\rho]) &= \mathscr{H}^3_{KM}(\varepsilon_P), \quad E^4_{MK}([d\rho]) = \mathscr{H}^4_{KM}(\varepsilon_P). \end{aligned}$$

Proof. The proof is based on results, given by Lemma 6, Lemma 7, and direct calculations.

Remark 3. In this remark we briefly discuss some global aspects of the theory of variational sequences on velocity manifolds as considered in this work; complete discussion goes outside the scope of this paper. By definition, the Euler-Lagrange mapping as well as the Helmholtz mapping are morphisms of the variational sequence. Consequently, the concepts of the kernel and the image of these mappings are well defined, and one can determine the cohomology groups of the corresponding complex of global sections.

We have constructed the variational sequence (15) on the fibred manifold $\mathbb{R} \times Y$; thus, applying to this Cartesian product the Künneth formula, we see that the De Rham cohomology groups satisfy $H^k(\mathbb{R} \times Y) = H^k Y$.

Consider the Euler-Lagrange morphism. By exactness of the variational sequence, if the Helmholtz class of the exterior derivative of a 2-form ρ vanishes, $[d\rho] = 0$, then the class of ρ, $[\rho]$ coincides, locally, with the class $[d\eta]$ for a 1-form η. In other words, if the Helmholtz expressions (27) vanish, then expressions (26) are locally variational, i.e., are locally of the form (25). In this sense the Helmholtz expressions define *local variationality conditions*.

However, a locally variational form may not be globally variational; it may happen that it does not possess a *global* Lagrangian. It follows from the properties of the variational sequence that a sufficient (topological) condition for existence of a global Lagrangian is the vanishing of the second cohomology group of the underlying fibred manifold, that is,

$H^2(\mathbb{R}\times Y)=0$. Applying to this general result of the Künneth formula, we see that if a source form $\varepsilon=\varepsilon_P\omega^P\wedge dt$ is locally variational and in addition $H^2Y=0$, then ε has a *global* Lagrangian.

If for example $Y=\mathbb{R}^m$, then since $H^k\mathbb{R}^m=0$ for every k, $1\leq k\leq m$, local variationality for source forms on $Y=\mathbb{R}^m$ always implies global variationality. If Y is the sphere S^m, then $H^kS^m=0$ for every $m\geq 2$ and k, $1\leq k\leq m-1$; thus we again get $H^2S^m=0$ whenever $m\geq 3$. If $m=2$, then $H^2S^2=\mathbb{R}\neq 0$, thus local variationality does not imply global variationality. Similarly, if $Y=S^1\times S^1$ is the torus, we get since $H^2(S^1\times S^1)=\mathbb{R}\neq 0$. If Y is the Möbius strip or the Klein bottle, in both cases $H^2Y=0$, and local variationality automatically implies global variationality.

5 INVARIANT VARIATIONAL FUNCTIONALS

In this section we study parameter invariance of variational functionals, associated with 1-forms on manifold of *regular* velocities. For purpose of applications, we consider second order case. Let W_0 be an open subset of Y, and $W_0^2=(\tau^{2,0})^{-1}(W_0)\subset T^2Y$.

Consider a 1-form $\rho\in\Omega^2_{1,0}W$, defined on $W_0^2\cap \mathrm{Imm}T^2Y$. Recall that every diffeomorphism $\mu:J\to I$ of open intervals induces a curve $s\to\mu^2(s)$ in L^2, defined on J (4).

Let $\gamma:I\to W_0\subset Y$ be an immersion. Any *compact* subinterval K of I associates with 1-form ρ the *variational functional* ρ_K, defined by

$$\gamma\to\rho_K(\gamma)=\int_K(T^2\gamma)^*\rho \tag{28}$$

on the set immersions $\gamma:K\to W_0\subset Y$ of class C^2.

Lemma 8. *Let $\rho\in\Omega^2_{1,0}W$ be a 1-form, and $\gamma:I\to W_0\subset Y$ be an immersion. Let $\mu:J\to I$ be a diffeomorphism of open intervals such that $D\mu>0$ on J. The following conditions are equivalent:*

(a) *For any compact subinterval $K\subset I$, the variational functional ρ_K is invariant with respect to reparametrization by diffeomorphism μ,*

$$\rho_K(\gamma)=\rho_{\mu^{-1}(K)}(\gamma\circ\mu). \tag{29}$$

(b) ρ *satisfies*

$$(T^2\gamma)^*\rho=(\mu^{-1})^*(T^2(\gamma\circ\mu))^*\rho. \tag{30}$$

Proof. This equivalence condition is a direct consequence of the change of variables theorem for integrals. We show that (a) implies (b). Since $D\mu>0$ on J,

$$\rho_{\mu^{-1}(K)}(\gamma\circ\mu)=\int_{\mu^{-1}(K)}(T^2(\gamma\circ\mu))^*\rho=\int_K(\mu^{-1})^*(T^2(\gamma\circ\mu))^*\rho. \tag{31}$$

The condition (29) holds for any compact subinterval $K\subset I$, hence the integrands of $\rho_K(\gamma)$ and $\rho_{\mu^{-1}(K)}(\gamma\circ\mu)$ coincide which directly results in (30). The converse is now obvious.

The variational functional ρ_K is said to be *parameter-invariant*, if one of the equivalent conditions of Lemma 8 is satisfied for every diffeomorphism $\mu:J\to I$ of open intervals such that $D\mu>0$ on J, and for every immersion $\gamma:I\to Y$. Condition (29), satisfied for all μ, also means that the *variational integral* $\rho_K(\gamma)$ does *not* depend on parametrization.

Now we find the necessary and sufficient conditions for a 1-form ρ to associate a parameter-invariant variational functional.

Theorem 4. *Let (V,ψ), $\psi=(y^K)$, be a chart on Y such that $V\subset W_0$. Let* 1-form $\rho\in\Omega^2_{1,0}W$ *be expressed by means of the contact basis in L-adapted chart $(V^{2,L},\chi^{2,L})$,*

$$\rho=A_Ldw^L+A_\sigma\eta^\sigma+\dot{A}_L\dot{\eta}^L+A^1_\sigma\eta^\sigma_1+\ddot{A}_Ld\ddot{w}^L+A^2_\sigma dw^\sigma_2. \tag{32}$$

The following two conditions are equivalent:

(a) *Variational functional ρ_K is parameter-invariant.*

(b) *The coefficients of ρ satisfy: $\ddot{A}_L$ vanishes identically on $V^{2,L}$, and A_L, A^2_σ* do not depend on $\dot{w}^L$, $\ddot{w}^L$.

Proof. Suppose that ρ_K, associated with 1-form ρ (32) for a given compact subinterval K, satisfies (a). Thus, by definition, we assume ρ satisfies condition (30) for every diffeomorphism μ such that $D\mu>0$, and every immersion γ with values in $W_0\subset Y$. Applying formulas of the chart expression of r-jet prolongation of a curve (see Lemma 3 and Remark 1), we obtain after a direct calculation the expressions of $(T^2\gamma)^*\rho$ and $(\mu^{-1})^*(T^2(\gamma\circ\mu))^*\rho=(T^2(\gamma\circ\mu)\circ\mu^{-1})^*\rho$,

$$(T^2\gamma)^*\rho(t)=\left(A_L+\ddot{A}_L\frac{\dddot{w}^L}{\dot{w}^L}+A^2_\sigma w^\sigma_3\right)dw^L\circ T^3\gamma(t),$$

and

$$\begin{aligned}(\mu^{-1})^*(T^2(\gamma\circ\mu))^*\rho(t) \quad &= \quad \Big(A_L(T^2\gamma(t)\circ J^2_0\mu_{\mu^{-1}(t)})+A^2_\sigma(T^2\gamma(t)\circ J^2_0\mu_{\mu^{-1}(t)})w^\sigma_3\\ &+ \quad \ddot{A}_L(T^2\gamma(t)\circ J^2_0\mu_{\mu^{-1}(t)})\frac{\dddot{w}^L(\dot{a})^3+3\ddot{w}^L\ddot{a}\dot{a}+\dot{w}^L\dddot{a}}{\dot{a}\dot{w}^L}\Big)dw^L\circ T^3\gamma(t),\end{aligned}$$

where $\dot{a}$, $\ddot{a}$, and $\dddot{a}$ denotes coordinates of $J_0^2\mu_{\mu^{-1}(t)} \in L^2$. Condition (30) then already implies (b). To show the converse, it is sufficient to verify condition (30) for ρ with specific coefficients given by (b). However, this is immediate.

Theorem 4 can be directly restated as follows.

Corollary. *A* 1*-form ρ defines parameter-invariant variational functional ρ_K if and only if there exists a decomposition of ρ,*

$$\rho = \rho_0 + \rho_c, \tag{33}$$

where ρ_0 is projectable onto the quotient space $\operatorname{Imm} T^2Y/L^2$*, called the Grassmann fibration of Y, and ρ_c is a contact form on* $\operatorname{Imm} T^2Y$ *in sense of Lemma 4,* (b). *Then however,* (33) *means that the class of ρ coincides with the class of ρ_0.*

Note that the class of 1-form ρ in sense of contact forms from Lemma 4 differs from the class, computed in Lemma 6, as en element of $\Omega_1^3 W/\Theta_1^3 W$. However, it can be easily shown that these classes coincide after pull-back by means of canonical prolongation $T^3\gamma$.

Suppose $\rho \in \Omega_{1,0}^2 W$ is expressed in the contact basis by (32). Let $\gamma : I \to Y$ be an immersion of open interval I into Y such that $\gamma(I) \subset W_0$, and $T^2\gamma(I) \subset V^{2,L}$. Then the pull-back $(T^2\gamma)^*\rho$ of ρ has the following chart expression,

$$(T^2\gamma)^*\rho = (\mathscr{L}_L \circ T^3\gamma)dt, \tag{34}$$

where $\mathscr{L}_L$ is the L-associated Lagrange function, given by

$$\mathscr{L}_L = A_L \dot{w}^L + A_\sigma^2 w_3^\sigma \dot{w}^L + \ddot{A}_L \ddot{w}^L. \tag{35}$$

We note that the function $\mathscr{L}_L \circ T^3\gamma$ coincides, after the pull-back by means of $T^3\gamma$, with the coefficient of class of 1-form ρ (17) in the variational sequence .

Now, suppose that ρ defines the parameter-invariant functional ρ_K. Then $\mathscr{L}_L$ reduces to

$$\mathscr{L}_L = A_L \dot{w}^L + A_\sigma^2 w_3^\sigma \dot{w}^L, \tag{36}$$

with the coefficients A_L, A_σ^2, not depending on $\dot{w}^L$ and $\ddot{w}^L$. The variational integral (28) can be now written of the form

$$\rho_K(\Omega) = \int_K (\mathscr{L}_L \circ T^3\gamma)dt, \tag{37}$$

and depends on a *subset* $\Omega = \gamma(K)$ in Y only. From the form of the variational integral (29), it follows that the equations for *extremals* of $\rho_K(\Omega)$ are the *Euler-Lagrange equations*

$$\varepsilon_K(\mathscr{L}_L) = 0,$$

where $\varepsilon_K(\mathscr{L}_L)$ are the Euler-Lagrange expressions, given by (25).

It is the standard result that the neccessary and sufficient conditions for the variational integral (28) to be parameter-invariant, are the well-known *Zermelo conditions* (Zermelo [11] , McKiernan [7]). An auxiliary lemma simplifying the Zermelo conditions, we proved in [10], is the following.

Lemma 9. *A function $F = F(y^K, \dot{y}^K, \ddot{y}^K)$, defined on* $\operatorname{Imm} T^2Y$ *satisfies the Zermelo conditions*

$$\frac{\partial F}{\partial \dot{y}^K}\dot{y}^K + 2\frac{\partial F}{\partial \ddot{y}^K}\ddot{y}^K = F, \quad \frac{\partial F}{\partial \ddot{y}^K}\dot{y}^K = 0,$$

if and only if the function $G(w^L, \dot{w}^L, \ddot{w}^L, w^\sigma, w_1^\sigma, w_2^\sigma) = F(y^K, \dot{y}^K, \ddot{y}^K)$ satisfies

$$\frac{\partial G}{\partial \dot{w}^L}\dot{w}^L = G, \quad \frac{\partial G}{\partial \ddot{w}^L} = 0.$$

Hence, we immediately see that the function $\mathscr{L}_L$ (36) associates parameter-invariant variational functional.

Acknowledgements. The first author was supported by the Ministry of Education, Youth and Sports of the Czech Republic, Project CZ.1.07/2.3.00/30.0058 "Development of Research Teams at the University of Pardubice".

The second author acknowledges support of National Science Foundation of China (grant No. 10932002) and of the Czech Science Foundation (grant 201/09/1981).

REFERENCES

[1] D. R. Grigore and D. Krupka, Invariants of velocities and higher-order Grassmann bundles, *J. Geom. Phys.* **24** (1998) 244–264.

[2] D. Krupka, Variational sequences on finite order jet spaces, in: D. Krupka, J. Janyška (Eds.) Diff. Geom. Appl. Proc. Conf., Brno, Czechoslovakia, 1989, 236–254, World Scientific, Singapore (1990).

[3] D. Krupka, Variational sequences in mechanics, *Calc. Var.* **5** (1997) 557–583.

[4] D. Krupka, Global variational theory in fibred spaces, in: D. Krupka and D. J. Saunders (Eds.) *Handbook of Global Analysis*, pp. 727791. Elsevier: Amsterdam (2007).

[5] D. Krupka and M. Krupka, Jets and contact elements, in: D. Krupka (Ed.), Proc. of the Seminar on Differential Geometry, Math. Publ. Vol. 2, Silesian Univ. Opava, Czech Republic, 2000, pp. 39–85.

[6] D. Krupka and Z. Urban, Differential invariants of velocities and higher order Grassmann bundles, in: O. Kowalski, D. Krupka, O. Krupková, J. Slovák (Eds.), Diff. Geom. Appl., Proc. Conf., in Honour of L. Euler, Olomouc, August 2007, World Scientific, Singapore, 2008, pp. 463–473.

[7] M. A. McKiernan, Sufficiency of parameter invariance conditions in areal and higher order Kawaguchi spaces, *Publ. Math. Debrecen* **13** (1966) 77–85.

[8] Z. Urban and D. Krupka, Variational sequences in mechanics on Grassmann fibrations, *Acta Appl. Math.* **112** (2010) 225–249.

[9] Z. Urban, Variational sequences in mechanics on Grassmann fibrations, PhD. Thesis, University of Ostrava (2011).

[10] Z. Urban and D. Krupka, The Zermelo conditions and higher order homogeneous functions, *Math. Publ. Debrecen*, in print (2012).

[11] E. Zermelo, Untersuchungen zur Variationsrechnung, Dissertation, Friedrich-Wilhelms-Universität, Berlin, 1894.