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# Integrable Couplings, Variational Identities and Hamiltonian Formulations

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**ABSTRACT:** We discuss Hamiltonian formulations for integrable couplings, particularly bi- and tri-integrable couplings, based on zero curvature equations. The basic tools are the variational identities over non-semisimple Lie algebras consisting of block matrices. Illustrative examples include dark equations and bi- and tri-integrable couplings of the KdV equation and the AKNS equations, generated from the enlarged matrix spectral problems. The associated variational identities yield bi-Hamiltonian formulations and hereditary recursion operators, thereby showing their Liouville integrability.

## 1 INTRODUCTION

The discovery of solitons in the late 1960s (Zabusky & Kruskal, 1965) has inspired a modern resurgence of interest in integrable systems. Solitons are strongly stable and localized solutions of nonlinear partial differential equations like the Kortewegde Vries (KdV) equation (Korteweg & de Vries, 1895), which describes waves on shallow water surfaces. These equations can be viewed as infinite dimensional integrable Hamiltonian systems (Dickey, 2003). Their study prompts a very fruitful approach to integrability of Hamiltonian systems, the inverse scattering transform (often reducible to Riemann-Hilbert problems), through solving associated integral equations (Gardner et al., 1967; Zakharov & Shabat, 1972; Wadati, 1972). The approach is a non-linear analogue of the Fourier transform.

All Hamiltonian systems are divided into two categories: integrable ones and non-integrable ones. The mathematical theory behind integrability of Hamiltonian systems is rich and interesting. Birkhoff stated "When, however, one attempts to formulate a precise definition of integrability, many possibilities appear, each with a certain intrinsic theoretic interest" (Birkhoff, 1927). This report will discuss the so-called Liouville integrable Hamiltonian systems, "not forgetting the dictum of H. Poincaré, that a system of differential equations is only more or less integrable" (Birkhoff, 1927).

A system of differential equations or differential-difference equations is said to be integrable by quadratures if its solutions can be obtained after a finite number of steps involving algebraic operations and integration of given functions (Arnold, 1989). Given a Hamiltonian system of ordinary differential equations (ODEs) on a symplectic space, when is it integrable by quadratures? A first answer is the Liouville theorem (Bour, 1855; Liouville, 1855). It tells that for a Hamiltonian system of ODEs defined on a symplectic space $(M^{2N}, \omega^2, H)$:

$$p_t = \{p, H\} = \omega^2(IdH, Idp),\ q_t = \{q, H\} = \omega^2(IdH, Idq), \tag{1.1}$$

the following conditions guarantee integrability by quadratures:

- Existence of $N$ integrals of motion $\{F_i(p,q)\}_{i=1}^N$;
- $F_1, F_2, \cdots, F_N$ are in involution in pairs: $\{F_i, F_j\} = 0,\ 1 \le i, j \le N$;
- $F_1, F_2, \cdots, F_N$ are functionally independent on the intersection of their level sets.

Those conditions are called the Liouville conditions, and Liouville integrable systems of ODEs mean Hamiltonian systems possessing the above Liouville conditions.

However, there are no similar Liouville conditions for guaranteeing integrability by quadratures of partial differential equations (PDEs) or differential-difference equations (DDEs). We list a few integrable criteria for PDEs and DDEs widely adopted in the soliton community:

- Lax pair and inverse scattering transform (S-integrable case);
- Transform into linear equations (C-integrable case);
- Painlevé test and the singularity confinement;
- Three-soliton solutions (equations having 3-solitons are called soliton equations);
- Infinitely many symmetries;
- Infinitely many conservation laws;
- Bi-Hamiltonian formulation (which often implies the existence of infinitely many symmetries and conservation laws).

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Liouville integrable systems of PDEs or DDEs are Hamiltonian systems which possess infinitely many commuting conservation laws. The fundamental question in the integrability theory of PDEs and DDEs is how to explore Hamiltonian structures and infinitely many commuting conservation laws. Lax proposed an equivalent operator formulation, now called the Lax pair approach, for studying the KdV equation (Lax, 1968). A Lax pair formulation is equivalent to a zero curvature equation formulation, which can be depicted as follows:

$$\begin{array}{ccc} \phi_x = U(u,\lambda)\phi \ \text{ or } \ E\phi = U(u,\lambda)\phi & \Leftrightarrow & u_t = \Phi^n K_0[u] \\ \Updownarrow & \boxed{\begin{array}{r} u_t = K_0[u] \ \Leftrightarrow \ U_t - V_x + [U,V] = 0 \\ \text{or } \ U_t + UV - (EV)U = 0 \end{array}} & \Updownarrow \\ \text{spectral matrix } U & \Leftrightarrow & \text{recursion operator } \Phi. \end{array}$$

A continuous (or discrete) Hamiltonian system of PDEs or DDEs reads

$$u_t = K(u, u_x, \cdots) \ [\text{or } K(u, Eu, E^{-1}u, \cdots)] = J\frac{\delta \mathcal{H}}{\delta u}, \tag{1.2}$$

where $\frac{\delta}{\delta u}$ is the variational derivative, $J$ is a Hamiltonian operator and $\mathcal{H}$ is called a Hamiltonian functional. For a Hamiltonian system, there exists a relationship chain:

$$\text{Conserved functional} \mapsto \text{adjoint symmetry} \mapsto \text{symmetry} : \mathcal{I} \mapsto \frac{\delta \mathcal{I}}{\delta u} \mapsto J\frac{\delta \mathcal{I}}{\delta u}, \tag{1.3}$$

and a Lie algebra homomorphism $J\frac{\delta}{\delta u}$:

$$J\frac{\delta}{\delta u}\{\mathcal{I}_1, \mathcal{I}_2\} = [J\frac{\delta \mathcal{I}_1}{\delta u}, J\frac{\delta \mathcal{I}_2}{\delta u}], \tag{1.4}$$

where the Poisson bracket of functionals and the Lie bracket of vector fields are defined by

$$\{\mathcal{I}_1, \mathcal{I}_2\} = \int (\frac{\delta \mathcal{I}_1}{\delta u})^T J \frac{\delta \mathcal{I}_2}{\delta u}\, dx, \ [K, S] = K'(u)[S] - S'(u)[K], \tag{1.5}$$

in which $P'(u)[v]$ denotes the Gateaux derivative of an object $P = P(u, u_x, \cdots)$ with respect to $u$ along a direction $v$:

$$P'(u)[v] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} P(u + \varepsilon v, u_x + \varepsilon v_x, \cdots). \tag{1.6}$$

Therefore, the fundamental question on integrable PDEs and DDES is now the following: Given a system of evolution equations generated from a zero curvature equation

$$u_t = K(u) \ \Leftrightarrow \ U_t - V_x + [U,V] = 0 \ \ [\text{or } \ U_t + UV - (EV)U = 0], \tag{1.7}$$

how can we generate its Hamiltonian structure (1.2) with a recursion operator $\Phi$ (Olver, 1977) or even a bi-Hamiltonian structure (Magri, 1978):

$$u_t = K(u) = J\frac{\delta \mathcal{H}_1}{\delta u} = M\frac{\delta \mathcal{H}_2}{\delta u}, \tag{1.8}$$

which naturally yields infinitely many commuting conservation laws?

The continuous zero curvature equation

$$U_t - V_x + [U,V] = 0 \tag{1.9}$$

and the discrete zero curvature equation

$$U_t + UV - (EV)U = 0 \tag{1.10}$$

are key elements in formulating Hamiltonian integrable systems. The Lax pair $U = U(u,\lambda)$ and $V = V(u,\lambda)$, with $\lambda$ being the spectral parameter, are square matrices often belonging to semisimple matrix Lie algebras. Integrable couplings, generated from enlarged Lax pairs, are certain enlarged non-trivial integrable systems including the original system (1.2) as a subsystem (Ma & Fuchssteiner, 1996; Ma, 2000). Since the integrability of a system has nothing to do with any arrangement of equations in the system, we will focus on triangular integrable systems, within which an initially given system will be listed as the first subsystem.

An integrable coupling of (1.2) is, thus, a triangular integrable system of the following form (Ma & Fuchssteiner, 1996):

$$\left\{ \begin{array}{l} u_t = K(u), \\ v_t = S(u,v). \end{array} \right. \tag{1.11}$$

If $S$ is nonlinear with respect to the second dependent variable $v$, the integrable coupling (1.11) is called nonlinear. An example of integrable couplings is the first-order perturbation system (Ma & Fuchssteiner, 1996):

$$\left\{ \begin{array}{l} u_t = K(u), \\ v_t = K'(u)[v], \end{array} \right. \tag{1.12}$$

where $K'$ denotes the Gateaux derivative of $K$.

A bi-integrable coupling of a given integrable system (1.2) is an enlarged triangular integrable system of the following form:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u, u_1), \\ u_{2,t} = S_2(u, u_1, u_2); \end{cases} \tag{1.13}$$

and similarly, by a tri-integrable coupling, we mean an enlarged triangular integrable system of the following form:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u, u_1), \\ u_{2,t} = S_2(u, u_1, u_2), \\ u_{3,t} = S_3(u, u_1, u_2, u_3). \end{cases} \tag{1.14}$$

Integrable couplings corresponds to non-semisimple Lie algebras $\bar{g}$ (Ma et al., 2006a; Ma et al., 2006b), and such Lie algebras can be written as semi-direct sums (Frappat et al., 2000):

$$\bar{g} = g \Subset g_c,\ g \text{ - semisimple},\ g_c \text{ - solvable}. \tag{1.15}$$

The notion of semi-direct sums

$$\bar{g} = g \Subset g_c \tag{1.16}$$

means that the two Lie subalgebras $g$ and $g_c$ satisfy

$$[g, g_c] \subseteq g_c, \tag{1.17}$$

where $[g, g_c] = \{[A, B] \,|\, A \in g,\ B \in g_c\}$, with $[\cdot, \cdot]$ denoting the Lie bracket of $\bar{g}$. Obviously, $g_c$ is an ideal Lie sub-algebra of $\bar{g}$. The subscript $c$ indicates a contribution to the construction of coupling systems. We also require the closure property between $g$ and $g_c$ under the matrix multiplication

$$g g_c, g_c g \subseteq g_c, \tag{1.18}$$

where $g_1 g_2 = \{AB \,|\, A \in g_1,\ B \in g_2\}$, while we use the discrete zero curvature equation over semi-direct sums of matrix Lie algebras to generate discrete coupling systems (Ma et al., 2006b).

Integrable couplings provide insightful clues for classifying integrable systems with multi-components (Ma & Fuchssteiner, 1996; Ma, 2000). Continuous and discrete zero curvature equations over non-semisimple matrix Lie algebras are the basis for generating integrable couplings, and the associated classical and super variational identities offer fundamental tools to furnish their Hamiltonian structures (Ma, 2009; Ma, 2010).

In this report, we would like to explore non-semisimple Lie algebras consisting of block matrices, associated with which enlarged zero curvature equations generate dark equations and bi- and tri-integrable couplings, in both classical and super cases. Hamiltonian structures of the resulting coupling systems will be furnished through the variational identities on general Lie algebras (Ma & Chen, 2006; Ma et al., 2008).

More concretely, we will discuss classical and super variational identities, and dark equations and bi- and tri-integrable couplings. We will introduce matrix Lie algebras consisting of $2 \times 2$, $3 \times 3$ and $4 \times 4$ block matrices and apply them to the construction of integrable couplings, based on enlarged zero curvature equations. Applications will be made for the KdV equation and the AKNS soliton hierarchy as illustrative examples. Hamiltonian structures of the resulting dark equations and bi- and tri-integrable couplings will be presented through the associated variational identities. The presented matrix Lie algebras will be shown to be a starting point to construct integrable Hamiltonian couplings of given integrable systems.

# 2 CLASSICAL VARIATIONAL IDENTITIES

## 2.1 Variational Identities on Matrix Lie Algebras

Trace identities (Tu, 1989; Tu, 1990; Ma, 1992) state that if the underlying Lie algebra is semisimple, then under the Killing form, we have the continuous trace identity:

$$\frac{\delta}{\delta u} \int \text{tr}(W U_\lambda)\, dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{tr}(W \frac{\partial U}{\partial u}), \tag{2.1}$$

and the discrete trace identity:

$$\frac{\delta}{\delta u} \sum_{n \in \mathbb{Z}} \text{tr}(W U_\lambda) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{tr}(W \frac{\partial U}{\partial u}), \tag{2.2}$$

where $\gamma$ is a fixed constant, and $U$ and $W$, belonging to the underlying Lie algebra, satisfy the stationary continuous zero curvature equation

$$W_x = [U, W] \tag{2.3}$$

and the stationary discrete zero curvature equation

$$(EW)(EU) = UW, \tag{2.4}$$

respectively. Applications are made to typical integrable systems including the KdV equation, the AKNS equations, the Toda lattice equation and the Volterra lattice equation. One of the applications is to generate the Hamiltonian structure for the AKNS nonlinear Schrödinger equations:

$$u_t = \begin{bmatrix} p \\ q \end{bmatrix}_t = \begin{bmatrix} -\frac{1}{2}p_{xx} + p^2 q \\ \frac{1}{2}q_{xx} - pq^2 \end{bmatrix} = J\frac{\delta \mathcal{H}}{\delta u}, \tag{2.5}$$

where

$$J = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix},\ \mathcal{H} = \int [-\frac{1}{4}p^2q^2 + \frac{1}{12}(p_{xx}q - p_xq_x + pq_{xx})]\,dx, \tag{2.6}$$

from the spectral problem

$$\phi_x = U\phi,\ U = U(u,\lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix}. \tag{2.7}$$

If the underlying Lie algebra is non-semisimple, we have the continuous variational identity (Ma & Chen, 2006):

$$\frac{\delta}{\delta u}\int \langle W, U_\lambda\rangle\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\langle W, \frac{\partial U}{\partial u}\rangle, \tag{2.8}$$

and the discrete variational identity (Ma, 2007):

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}} \langle W, U_\lambda\rangle = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\langle W, \frac{\partial U}{\partial u}\rangle, \tag{2.9}$$

where $\gamma$ is a fixed constant, $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form, and $U$ and $W$, belonging to a Lie algebra, either semisimple or non-semisimple, satisfy (2.3) and (2.4), respectively. There are also explicit formulas for computing the involved constant $\gamma$.

- **The continuous case:** Assume that $W$ solves (2.3), i.e., $W_x = [U, W]$. If $|\langle W, W\rangle| \neq 0$, then

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|. \tag{2.10}$$

- **The discrete case:** Assume that $W$ solves (2.4), i.e., $(EW)(EU) = UW$ and let $\Gamma = WU$. If $|\langle\Gamma,\Gamma\rangle| \neq 0$, then

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle\Gamma,\Gamma\rangle|. \tag{2.11}$$

If the underlying Lie algebra is simesimple, then all non-degenerate, symmetric and ad-invariant bilinear forms are equivalent to the Killing form up to a constant multiplier. The Killing form on a Lie algebra is non-degenerate if and only if the Lie algebra is semisimple. Therefore, we have to use the variational identities to construct Hamiltonian structures of soliton hierarchies (see, e.g., Xia et al., 2004; Zhang, 2004; Fan & Zhang, 2005; Li & Dong, 2008; Feng & Liu, 2011). Nevertheless, the above variational identities need to be generalized to establish Hamiltonian formulations associated with more general non-semisimple Lie algebras. So far, we do not know if there is any Hamiltonian structure for the following bi-integrable coupling:

$$\begin{cases} u_t = K(u), \\ v_t = K'(u)[v], \\ w_t = K'(u)[w]. \end{cases} \tag{2.12}$$

## 2.2 Construction of Bilinear Forms

To construct non-degenerate, symmetric and ad-invariant bilinear forms on the underlying matrix Lie algebra, let us say, $\bar{g}$, conveniently, we first transform the underlying matrix Lie algebra $\bar{g}$ into a vector form. Define a mapping

$$\sigma : \bar{g} \to \mathbb{R}^k,\ A \mapsto (a_1, \cdots, a_k)^T, \tag{2.13}$$

where $k$ is the dimension of the underlying matrix Lie algebra $\bar{g}$. This mapping $\sigma$ induces a Lie algebraic structure on $\mathbb{R}^k$, isomorphic to the matrix Lie algebra $\bar{g}$. The corresponding Lie bracket $[\cdot,\cdot]$ on $\mathbb{R}^k$ can be computed as follows

$$[a, b]^T = a^T R(b),\ a = (a_1, \cdots, a_k)^T,\ b = (b_1, \cdots, b_k)^T \in \mathbb{R}^k, \tag{2.14}$$

where $R(b)$ is the unique matrix generated from the Lie bracket $[a, b]$. This Lie algebra $(\mathbb{R}^k, [\cdot,\cdot])$ is isomorphic to the underlying matrix Lie algebra $\bar{g}$, and the mapping $\sigma$, defined by (2.13), is a Lie algebra isomorphism between the two Lie algebras.

Now, take an arbitrary bilinear form on $\mathbb{R}^k$:

$$\langle a, b\rangle = a^T F b, \tag{2.15}$$

where $F$ is a constant matrix (actually, $F = (\langle \mathbf{e}_i, \mathbf{e}_j\rangle)_{k\times k}$, where $\mathbf{e}_1, \cdots, \mathbf{e}_k$ are the standard basis of $\mathbb{R}^k$). The symmetric property

$$\langle a, b\rangle = \langle b, a\rangle \tag{2.16}$$

implies that

$$F^T = F. \tag{2.17}$$

Under this symmetric condition, the ad-invariance property

$$\langle a, [b, c] \rangle = \langle [a, b], c \rangle \tag{2.18}$$

becomes equivalent to the property

$$F(R(b))^T = -R(b)F,\ b \in \mathbb{R}^k. \tag{2.19}$$

This matrix equation with an arbitrary $b$ engenders a linear system of equations on the elements of the matrix $F$. Solving the resulting system tells what the matrix $F$ will be. Normally, $F$ contains a few arbitrary constants.

Then, the corresponding bilinear form on the underlying matrix Lie algebra $\bar{g}$ is given by

$$\langle A, B \rangle_{\bar{g}} \quad = \quad \langle \sigma(A), \sigma(B) \rangle_{\mathbb{R}^k} = (a_1, \cdots, a_k) F (b_1, \cdots, b_k)^T, \tag{2.20}$$

where

$$A = \sigma^{-1}((a_1, \cdots, a_k)^T) \in \bar{g},\ B = \sigma^{-1}((b_1, \cdots, b_k)^T) \in \bar{g}. \tag{2.21}$$

Owing to the isomorphism of $\sigma$, the bilinear form (2.20) is also symmetric and ad-invariant:

$$\langle A, B \rangle_{\bar{g}} = \langle B, A \rangle_{\bar{g}},\ \langle A, [B, C] \rangle_{\bar{g}} = \langle [A, B], C \rangle_{\bar{g}},\ A, B, C \in \bar{g}. \tag{2.22}$$

This kind of bilinear forms is not of Killing type, if the underlying matrix Lie algebra $\bar{g}$ is non-semisimple. The bilinear form (2.20) is non-degenerate if and only if the determinant of $F$ is not zero. Therefore, we can choose values for the constants in $F$ such that det$(F)$ is non-zero to get a non-degenerate bilinear form required in the variational identities.

# 3 MATRIX LIE ALGEBRAS AND INTEGRABLE COUPLINGS

## 3.1 Lie Algebras

In order to generate integrable couplings, one needs to create associated Lie algebras (see, e.g., Guo & Zhang, 2003; Xia et al., 2004). One of the ways to do it is to develop matrix Lie algebras consisting of block matrices (Ma, 2012). In this report, we particularly develop matrix Lie algebras consisting of $2 \times 2$, $3 \times 3$ or $4 \times 4$ block matrices, to generate dark equations and bi- and tri-integrable couplings. Larger numbers of blocks bring complexity in theoretical verification.

**Class 1 - Matrix Lie algebras consisting of $2 \times 2$ block matrices:**

All $2 \times 2$ block matrices of the following type:

$$M_1(A_1, A_2) = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1 \end{bmatrix}, \tag{3.1}$$

where $A_1$ and $A_2$ are square matrices of the same order, form a matrix Lie algebra with a semi-direct sum decomposition

$$\bar{g} = g \Subset g_c,\ \text{ where } g = \{M_1(A_1, 0)\},\ g_c = \{M_1(0, A_2)\}.$$

The corresponding matrix product reads

$$M_1(A_1, A_2) M_1(B_1, B_2) = M_1(C_1, C_2), \tag{3.2}$$

with $C_1$ and $C_2$ being defined by

$$\begin{cases} C_1 = A_1 B_1, \\ C_2 = A_1 B_2 + A_2 B_1. \end{cases} \tag{3.3}$$

This kind of Lie algebras will be used to generate dark equations, and the variational identity in this case reduces to the bi-trace identity (Ma & Zhang, 2010).

**Class 2 - Matrix Lie algebras consisting of $3 \times 3$ block matrices:**

Let $\alpha$ and $\beta$ be two arbitrarily given constants, which could be zero. All $3 \times 3$ block matrices of the following type:

$$M_2(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & \beta A_2 + \alpha A_3 \\ 0 & 0 & A_1 + \alpha A_2 \end{bmatrix}, \tag{3.4}$$

where $A_1$, $A_2$ and $A_3$ be square matrices of the same order, form a matrix Lie algebra with a semi-direct sum decomposition

$$\bar{g} = g \Subset g_c,\ \text{ where } g = \{M_2(A_1, 0, 0)\},\ g_c = \{M_2(0, A_2, A_3)\}.$$

The corresponding matrix product reads

$$M_2(A_1, A_2, A_3) M_2(B_1, B_2, B_3) = M_2(C_1, C_2, C_3), \tag{3.5}$$

with $C_1, C_2$ and $C_3$ being defined by

$$\begin{cases} C_1 = A_1B_1, \\ C_2 = A_1B_2 + A_2B_1 + \alpha A_2B_2, \\ C_3 = A_1B_3 + A_3B_1 + \beta A_2B_2 + \alpha A_2B_3 + \alpha A_3B_2. \end{cases} \tag{3.6}$$

This kind of Lie algebras will be used to generate bi-integrable couplings.

**Class 3 - Matrix Lie algebras consisting of $4 \times 4$ block matrices:**

Let $\alpha, \beta, \mu$ and $\nu$ be four arbitrarily given constants, which could be zero. The following $4 \times 4$ block matrices:

$$M_3(A_1, A_2, A_3, A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 + \alpha A_2 & \alpha A_3 & \beta A_2 + \alpha A_4 \\ 0 & 0 & A_1 + \alpha A_2 + \mu A_3 & \nu A_3 \\ 0 & 0 & 0 & A_1 + \alpha A_2 \end{bmatrix}, \tag{3.7}$$

where $A_i$, $1 \le i \le 4$, be square matrices of the same order, form a matrix Lie algebra with a semi-direct sum decomposition

$$\bar{g} = g \Subset g_c, \text{ where } g = \{M_3(A_1, 0, 0, 0)\}, \; g_c = \{M_3(0, A_2, A_3, A_4)\}.$$

The corresponding matrix product reads

$$M_3(A_1, A_2, A_3, A_4)M_3(B_1, B_2, B_3, B_4) = M_3(C_1, C_2, C_3, C_4), \tag{3.8}$$

with $C_1, C_2, C_3$ and $C_4$ being defined by

$$\begin{cases} C_1 = A_1B_1, \\ C_2 = A_1B_2 + A_2B_1 + \alpha A_2B_2, \\ C_3 = A_1B_3 + A_3B_1 + \alpha A_2B_3 + \alpha A_3B_2 + \mu A_3B_3, \\ C_4 = A_1B_4 + A_4B_1 + \alpha A_2B_4 + \alpha A_4B_2 + \beta A_2B_2 + \nu A_3B_3. \end{cases} \tag{3.9}$$

This kind of Lie algebras will be used to generate tri-integrable couplings (Ma et al., 2012).

## 3.2 Dark Equations

Dark equations (see, e.g., Ma, 2010) are given by

$$\begin{cases} u_t = K(u), \\ \psi_t = A(u, \partial_x)\psi, \end{cases} \tag{3.10}$$

where $A$ is a linear differential operator.

An example is the KdV perturbation system (Ma & Fuchssteiner, 1996):

$$\begin{cases} u_t = 6uu_x + u_{xxx}, \\ \psi_t = 6(u\psi)_x + \psi_{xxx}. \end{cases} \tag{3.11}$$

The trace identity yields the Hamiltonian structure of the KdV equation:

$$u_t = 6uu_x + u_{xxx} = J\frac{\delta \mathcal{H}}{\delta u}, \text{ where } J = \partial, \; \mathcal{H} = \int (-\frac{u_x^2}{2} + u^3)\, dx. \tag{3.12}$$

An application of the bi-trace identity associated with a perturbation spectral matrix

$$\bar{U} = \begin{bmatrix} U(u) & U'(u)[v] \\ 0 & U(u) \end{bmatrix} \tag{3.13}$$

determines the Hamiltonian structure of the perturbation system (3.11):

$$\bar{u}_t = \begin{bmatrix} u \\ \psi \end{bmatrix}_t = \bar{J}\frac{\delta \bar{\mathcal{H}}}{\delta \bar{u}}, \text{ where } \bar{J} = \begin{bmatrix} 0 & \partial \\ \partial & 0 \end{bmatrix}, \; \bar{\mathcal{H}} = \int (-u_x\psi_x + 3u^2\psi)\, dx. \tag{3.14}$$

Another direct application of the bi-trace identity associated with (3.13) presents the Hamiltonian dark equations of the AKNS nonlinear Schrödinger equations:

$$\bar{u}_t = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}_t = \begin{bmatrix} -\frac{1}{2}p_{xx} + p^2q \\ \frac{1}{2}q_{xx} - pq^2 \\ -\frac{1}{2}r_{xx} + 2pqr + p^2s \\ \frac{1}{2}s_{xx} - q^2r - 2pqs \end{bmatrix} = \bar{J}\frac{\delta \bar{\mathcal{H}}}{\delta \bar{u}}, \tag{3.15}$$

where $\bar J$ and $\bar{\mathcal{H}}$ are given by

$$\bar J = \left[\begin{array}{cc} 0 & J \\ J & -J \end{array}\right],\ \bar{\mathcal{H}} = \int [-\frac{1}{2}(pq^2r + p^2qs) + \frac{1}{12}(p_{xx}s + q_{xx}r - p_xs_x - q_xr_x + qr_{xx} + ps_{xx})]\,dx, \quad (3.16)$$

with $J$ being defined in (2.6).

A class of nonlinear integrable couplings can be similarly defined by

$$\begin{cases} u_t = K(u), \\ \psi_t = A(u,\partial_x)\psi + B(u,\partial_x)\psi^2, \end{cases} \quad (3.17)$$

where $A$ and $B$ are linear differential operators. But it is not yet clear to us how to find their Hamiltonian structures.

## 3.3 Bi- and Tri-Integrable Couplings

We start from the Lie algebras (3.4) and (3.7) to generate a kind of bi- and tri-integrable Hamiltonian couplings for the AKNS soliton hierarchy, respectively.

### 3.3.1 The AKNS Soliton Hierarchy

Let us recall the AKNS soliton hierarchy (Ablowitz et al., 1974). The traditional spectral problem for the AKNS hierarchy is defined by

$$\phi_x = U\phi,\ U = U(u,\lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix},\ u = \begin{bmatrix} p \\ q \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \quad (3.18)$$

The stationary zero curvature equation (2.3), i.e., $W_x = [U, W]$, is equivalent to

$$\begin{cases} a_x = pc - qb, \\ b_x = -2\lambda\, b - 2\,pa, \\ c_x = 2qa + 2\lambda\, c, \end{cases} \quad (3.19)$$

if we assume that $W$ is of the form

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\ge 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}. \quad (3.20)$$

The system (3.19) equivalently requires

$$\begin{cases} b_{i+1} = -\frac{1}{2}b_{i,x} - pa_i, \\ c_{i+1} = \frac{1}{2}c_{i,x} - qa_i, \qquad i \ge 0, \\ a_{i+1,x} = pc_{i+1} - qb_{i+1}, \end{cases} \quad (3.21)$$

upon taking the initial values

$$a_0 = -1,\ b_0 = c_0 = 0. \quad (3.22)$$

We impose the integration conditions

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0,\ i \ge 1, \quad (3.23)$$

such that the recursion relations in (3.21) will uniquely determine the sequence of $\{a_i, b_i, c_i |\, i \ge 1\}$. Then, the first few sets are given by

$$\begin{array}{l} b_1 = p,\ c_1 = q,\ a_1 = 0; \\ b_2 = -\frac{1}{2}p_x,\ c_2 = \frac{1}{2}q_x,\ a_2 = \frac{1}{2}pq; \\ b_3 = \frac{1}{4}p_{xx} - \frac{1}{2}p^2q,\ c_3 = \frac{1}{4}q_{xx} - \frac{1}{2}pq^2,\ a_3 = \frac{1}{4}(pq_x - p_xq). \end{array}$$

Now, the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0 \text{ with } V^{[m]} = (\lambda^m W)_+,\ \ m \ge 0, \quad (3.24)$$

where $P_+$ denotes the polynomial part of $P$ in $\lambda$, generate the AKNS soliton hierarchy:

$$u_{t_m} = K_m = \left[\begin{array}{c} -2b_{m+1} \\ 2c_{m+1} \end{array}\right] = \Phi^m \left[\begin{array}{c} -2p \\ 2q \end{array}\right] = J\frac{\delta \mathcal{H}_m}{\delta u},\ \ m \ge 0, \quad (3.25)$$

where the Hamiltonian operator, the hereditary recursion operator and the Hamiltonian functionals are given by

$$J = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix},\ \ \Phi = \left[\begin{array}{cc} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{array}\right],\ \ \mathcal{H}_m = \int \frac{2a_{m+2}}{m+1}\,dx,\ m \ge 0, \quad (3.26)$$

respectively.

### 3.3.2 Bi-Integrable Couplings

We use the class of matrix Lie algebras defined by (3.4), and begin with an enlarged spectral matrix

$$\bar{U} = \bar{U}(\bar{u},\lambda) = M_2(U,U_1,U_2),\ \bar{u} = (p,q,r,s,v,w)^T, \tag{3.27}$$

where $U$ is defined as in (3.18) and the supplementary spectral matrices $U_1$ and $U_2$ read

$$\begin{cases} U_1 = U_1(u_1) = \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix},\ u_1 = \begin{bmatrix} r \\ s \end{bmatrix}, \\ U_2 = U_2(u_2) = \begin{bmatrix} 0 & v \\ w & 0 \end{bmatrix},\ u_2 = \begin{bmatrix} v \\ w \end{bmatrix}. \end{cases} \tag{3.28}$$

To solve the enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U},\bar{W}], \tag{3.29}$$

we take a solution of the following type

$$\bar{W} = \bar{W}(\bar{u},\lambda) = M_2(W,W_1,W_2) \tag{3.30}$$

where $W$, defined by (3.20), solves (2.3), i.e., $W_x = [U,W]$, and $W_1$ and $W_2$ read

$$\begin{cases} W_1 = W_1(u,u_1,\lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix}, \\ W_2 = W_2(u,u_1,u_2,\lambda) = \begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix}. \end{cases} \tag{3.31}$$

Then, besides (2.3), the enlarged stationary zero curvature equation (3.29) equivalently generates

$$\begin{cases} e_x = (p+\alpha r)g - (q+\alpha s)f + rc - sb, \\ f_x = -2\lambda f - 2(p+\alpha r)e - 2ra, \\ g_x = 2(q+\alpha s)e + 2\lambda g + 2sa, \end{cases}$$

and

$$\begin{cases} e'_x = (p+\alpha r)g' - (q+\alpha s)f' + (\beta r+\alpha v)g - (\beta s+\alpha w)f + vc - wb, \\ f'_x = -2\lambda f' - 2(p+\alpha r)e' - 2(\beta r+\alpha v)e - 2va, \\ g'_x = 2(q+\alpha s)e' + 2\lambda g' + 2(\beta s+\alpha w)e + 2wa. \end{cases}$$

Trying a formal series solution $\bar{W}$ by assuming

$$\begin{cases} e = \sum_{i=0}^{\infty} e_i\lambda^{-i},\ f = \sum_{i=0}^{\infty} f_i\lambda^{-i},\ g = \sum_{i=0}^{\infty} g_i\lambda^{-i}, \\ e' = \sum_{i=0}^{\infty} e'_i\lambda^{-i},\ f' = \sum_{i=0}^{\infty} f'_i\lambda^{-i},\ g' = \sum_{i=0}^{\infty} g'_i\lambda^{-i}, \end{cases} \tag{3.32}$$

we arrive at

$$\begin{cases} f_{i+1} = -\frac{1}{2}f_{i,x} - (p+\alpha r)e_i - ra_i, \\ g_{i+1} = \frac{1}{2}g_{i,x} - (q+\alpha s)e_i - sa_i, \\ e_{i+1,x} = (p+\alpha r)g_{i+1} - (q+\alpha s)f_{i+1} + rc_{i+1} - sb_{i+1}, \end{cases} \tag{3.33}$$

and

$$\begin{cases} f'_{i+1} = -\frac{1}{2}f'_{i,x} - (p+\alpha r)e'_i - (\beta r+\alpha v)e_i - va_i, \\ g'_{i+1} = \frac{1}{2}g'_{i,x} - (q+\alpha s)e'_i - (\beta s+\alpha w)e_i - wa_i, \\ e'_{i+1,x} = (p+\alpha r)g'_{i+1} - (q+\alpha s)f'_{i+1} + (\beta r+\alpha v)g_{i+1} - (\beta s+\alpha w)f_{i+1} + vc_{i+1} - wb_{i+1}, \end{cases} \tag{3.34}$$

where $i \geq 0$. We select the initial data to be

$$e_0 = -1,\ f_0 = g_0 = 0;\ e'_0 = -1,\ f'_0 = g'_0 = 0; \tag{3.35}$$

and impose that

$$e_i|_{\bar{u}=0} = f_i|_{\bar{u}=0} = g_i|_{\bar{u}=0} = 0,\ e'_i|_{\bar{u}=0} = f'_i|_{\bar{u}=0} = g'_i|_{\bar{u}=0} = 0,\ i \geq 1. \tag{3.36}$$

Then the recursion relations in (3.33) and (3.34) uniquely determine the sequences of $\{e_i, f_i, g_i | i \geq 1\}$ and $\{e'_i, f'_i, g'_i | i \geq 1\}$. We point out that it is easy to check that first few sets of $\{a_i, b_i, c_i | i \geq 1\}$, $\{e_i, f_i, g_i | i \geq 1\}$ and $\{e'_i, f'_i, g'_i | i \geq 1\}$ are all differential polynomials in six variables $p, q, r, s, v, w$.

**Integrable Couplings:**

Let us further define a sequence of enlarged Lax matrices

$$\bar{V}^{[m]} = (\lambda^m \bar{W})_+ = M_2(V^{[m]}, V_1^{[m]}, V_2^{[m]}),\ m \geq 0, \tag{3.37}$$

where $V^{[m]}$ is defined as in (3.24) and $V_i^{[m]} = (\lambda^m W_i)_+,\ i = 1, 2$, and then, the enlarged zero curvature equations

$$\bar{U}_{t_m} - (\bar{V}^{[m]})_x + [\bar{U}, \bar{V}^{[m]}] = 0,\ m \geq 0, \tag{3.38}$$

generate a hierarchy of bi-integrable couplings:

$$\bar{u}_{t_m} = \begin{bmatrix} p \\ q \\ r \\ s \\ v \\ w \end{bmatrix}_{t_m} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \end{bmatrix} = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \\ -2f'_{m+1} \\ 2g'_{m+1} \end{bmatrix},\ m \geq 0, \tag{3.39}$$

for the AKNS hierarchy (3.25).

Except the first two, all bi-integrable couplings presented above are nonlinear, since the supplementary systems with $m \geq 2$ are nonlinear with respect to the four dependent variables $r, s, v, w$. This implies that (3.39) provides a hierarchy of nonlinear bi-integrable couplings for the AKNS soliton hierarchy (3.25). The first nonlinear bi-integrable coupling system is given by

$$p_{t_2} = -2b_3,\ q_{t_2} = 2c_3,\ r_{t_2} = -2f_3,\ s_{t_2} = 2g_3,\ v_{t_2} = -2f'_3,\ w_{t_2} = 2g'_3, \tag{3.40}$$

where $b_3, c_3, f_3, g_3, f'_3, g'_3$ are defined before.

**Hamiltonian structures:**

To furnish Hamiltonian structures of the obtained bi-integrable couplings, we need to compute non-degenerate, symmetric and ad-invariant bilinear forms on the adopted matrix loop algebra. Following the general procedure in subsection 2.2, a direct computation tells

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\,\eta_2 + \beta\,\eta_3 & \alpha\,\eta_3 \\ \eta_3 & \alpha\,\eta_3 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{3.41}$$

where $\eta_i$, $1 \leq i \leq 3$, are arbitrary constants and $\otimes$ is the Kronecker product. Therefore, a required bilinear form on the underlying Lie algebra is determined by

$$\begin{aligned} \langle A, B \rangle &= \langle \sigma(A), \sigma(B) \rangle_{\mathbb{R}^9} = (a_1, \cdots, a_9) F (b_1, \cdots, b_9)^T \\ &= (2\,a_1 b_1 + a_2 b_3 + a_3 b_2)\eta_1 \\ &\quad + (2\,a_1 b_4 + a_2 b_6 + a_3 b_5 + 2\,a_4 b_1 + 2\,\alpha\,a_4 b_4 \\ &\quad + a_5 b_3 + \alpha\,a_5 b_6 + a_6 b_2 + \alpha\,a_6 b_5)\eta_2 \\ &\quad + (2\,a_1 b_7 + a_2 b_9 + a_3 b_8 + 2\,\beta\,a_4 b_4 + 2\,\alpha\,a_4 b_7 \\ &\quad + \alpha\,a_5 b_9 + \beta\,a_5 b_6 + \beta\,a_6 b_5 + \alpha\,a_6 b_8 + 2\,a_7 b_1 \\ &\quad + 2\,\alpha\,a_7 b_4 + \alpha\,a_8 b_6 + a_8 b_3 + a_9 b_2 + \alpha\,a_9 b_5)\eta_3, \end{aligned} \tag{3.42}$$

where $A = A(a_1, a_2, \cdots, a_9) = M_2(A_1, A_2, A_3)$ and $B = B(b_1, b_2, \cdots, b_9) = M_2(B_1, B_2, B_3)$ are two block matrices of the form (3.4) with the blocks

$$A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix},\ B_i = \begin{bmatrix} b_{3i-2} & b_{3i-1} \\ b_{3i} & -b_{3i-2} \end{bmatrix},\ 1 \leq i \leq 3. \tag{3.43}$$

This bilinear form (3.42) is symmetric and ad-invariant:

$$\langle A, B \rangle = \langle B, A \rangle,\ \langle A, [B, C] \rangle = \langle [A, B], C \rangle,$$

and it is non-degenerate if and only if

$$\det(F) = 8\,(\alpha^2 \eta_1 - \alpha\,\eta_2 + \beta\,\eta_3)^3 {\eta_3}^6 \neq 0. \tag{3.44}$$

To apply the variational identity, let us further compute that

$$\langle \bar W, \bar U_\lambda\rangle = -2\,a\eta_1 - 2\,e\eta_2 - 2\,e'\eta_3,$$

and

$$\langle \bar W, \bar U_{\bar u}\rangle = \begin{bmatrix} c\,\eta_1 + g\,\eta_2 + g'\eta_3 \\ b\,\eta_1 + f\,\eta_2 + f'\eta_3 \\ (c+\alpha\,g)\,\eta_2 + (\beta\,g + \alpha\,g')\,\eta_3 \\ (b+\alpha\,f)\,\eta_2 + (\beta\,f + \alpha\,f')\,\eta_3 \\ (c+\alpha\,g)\,\eta_3 \\ (b+\alpha\,f)\,\eta_3 \end{bmatrix},$$

where $\bar W$ is given by (3.30). Therefore, the corresponding variational identity with $\gamma = 0$ yields the following Hamiltonian structures for the AKNS bi-integrable couplings in (3.39):

$$\bar u_{t_m} = \bar K_m(\bar u) = \bar J \frac{\delta \bar{\mathcal H}_m}{\delta \bar u},\ m \ge 0, \tag{3.45}$$

where the Hamiltonian operator is

$$\bar J = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\,\eta_2 + \beta\,\eta_3 & \alpha\,\eta_3 \\ \eta_3 & \alpha\,\eta_3 & 0 \end{bmatrix}^{-1} \otimes \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \tag{3.46}$$

and the Hamiltonian functionals read

$$\bar{\mathcal H}_m = \int \frac{2\eta_1 a_{m+2} + 2\eta_2 e_{m+2} + 2\eta_3 e'_{m+2}}{m+1}\,dx,\ m \ge 0. \tag{3.47}$$

**Recursion operator:**
Based on (3.21), (3.33) and (3.34), a direct computation yields a recursion relation

$$\bar K_{m+1} = \bar\Phi \bar K_m,\ m \ge 1, \tag{3.48}$$

where the recursion operator $\bar\Phi$ is given by

$$\bar\Phi = \begin{bmatrix} \Phi & 0 & 0 \\ \Phi_1 & \Phi + \alpha\Phi_1 & 0 \\ \Phi_2 & \beta\Phi_1 + \alpha\Phi_2 & \Phi + \alpha\Phi_1 \end{bmatrix}, \tag{3.49}$$

with $\Phi$ being given as in (3.26) and $\Phi_1$ and $\Phi_2$ being defined by

$$\Phi_1 = \begin{bmatrix} r\partial^{-1}q + (p+\alpha r)\partial^{-1}s & r\partial^{-1}p + (p+\alpha r)\partial^{-1}r \\ -s\partial^{-1}q - (q+\alpha s)\partial^{-1}s & -s\partial^{-1}p - (q+\alpha s)\partial^{-1}r \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} v\partial^{-1}q + (\beta r + \alpha v)\partial^{-1}s + (p+\alpha r)\partial^{-1}w & v\partial^{-1}p + (\beta r + \alpha v)\partial^{-1}r + (p+\alpha r)\partial^{-1}v \\ -w\partial^{-1}q - (\beta s + \alpha w)\partial^{-1}s - (q+\alpha s)\partial^{-1}w & -w\partial^{-1}p - (\beta s + \alpha w)\partial^{-1}r - (q+\alpha s)\partial^{-1}r \end{bmatrix}.$$

### 3.3.3 Tri-Integrable Couplings

To construct tri-integrable couplings for the AKNS equations, we use the class of matrix Lie algebras defined by (3.7), and begin with an enlarged spectral matrix

$$\bar U = \bar U(\bar u, \lambda) = M_3(U, U_1, U_2, U_3),\ u = (u^T, u_1^T, u_2^T, u_3^T)^T, \tag{3.50}$$

where $U = U(u,\lambda)$ is defined as in (3.18) and the supplementary spectral matrices and the new dependent variables are given by

$$U_i = U_i(u_i) = \begin{bmatrix} 0 & r_i \\ s_i & 0 \end{bmatrix},\ u_i = \begin{bmatrix} r_i \\ s_i \end{bmatrix},\ 1 \le i \le 3. \tag{3.51}$$

To solve the corresponding enlarged stationary zero curvature equation (3.29), we search for solutions of the following form

$$\bar W = \bar W(\bar u, \lambda) = M_3(W, W_1, W_2, W_3) \in \bar g, \tag{3.52}$$

where $W$ is given by (3.20). Then, the enlarged stationary zero curvature equation (3.29) gives

$$\begin{cases} W_{1,x} = [U, W_1] + [U_1, W] + \alpha\,[U_1, W_1], \\ W_{2,x} = [U, W_2] + \alpha\,[U_1, W_2] + [U_2, W] + \alpha\,[U_2, W_1] + \mu\,[U_2, W_2], \\ W_{3,x} = [U, W_3] + \beta\,[U_1, W_1] + \alpha\,[U_1, W_3] + \nu\,[U_2, W_2] + [U_3, W] + \alpha\,[U_3, W_1]. \end{cases} \tag{3.53}$$

Assume that $W_1, W_2, W_3$ are of the form

$$\left\{\begin{array}{l} W_1 = W_1(u,u_1,\lambda) = \left[\begin{array}{cc} e & f \\ g & -e \end{array}\right], \\ W_2 = W_2(u,u_1,u_2,\lambda) = \left[\begin{array}{cc} e' & f' \\ g' & -e' \end{array}\right], \\ W_3 = W_3(u,u_1,u_2,u_3,\lambda) = \left[\begin{array}{cc} e'' & f'' \\ g'' & -e'' \end{array}\right]. \end{array}\right. \tag{3.54}$$

The three equations in the above system (3.53) equivalently engender

$$\left\{\begin{array}{l} e_x = -s_1 b + r_1 c - (q+\alpha\, s_1) f + (p+\alpha\, r_1) g, \\ f_x = -2r_1 a - 2(p+\alpha\, r_1)e - 2\lambda\, f, \\ g_x = 2s_1 a + 2(q+\alpha\, s_1)e + 2\lambda\, g; \end{array}\right.$$

$$\left\{\begin{array}{l} e'_x = -s_2 b + r_2 c - (q+\alpha\, s_1+\mu\, s_2) f' + (p+\alpha\, r_1+\mu\, r_2) g' - \alpha\, s_2 f + \alpha\, r_2 g, \\ f'_x = -2r_2 a - 2(p+\alpha\, r_1+\mu\, r_2)e' - 2\lambda\, f' - 2\alpha\, r_2 e, \\ g'_x = 2s_2 a + 2(q+\alpha\, s_1+\mu\, s_2)e' + 2\lambda\, g' + 2\alpha\, s_2 e; \end{array}\right.$$

and

$$\left\{\begin{array}{rcl} e''_x &=& -s_3 b + r_3 c - (\beta\, s_1+\alpha\, s_3) f + (\beta\, r_1+\alpha\, r_3) g - \nu\, s_2 f' \\ && +\nu\, r_2 g' - (q+\alpha\, s_1) f'' + (p+\alpha\, r_1) g'', \\ f''_x &=& -2\lambda\, f'' - 2(\alpha\, r_3+\beta\, r_1)e - 2\nu\, r_2 e' - 2(p+\alpha\, r_1)e'' - 2r_3 a, \\ g''_x &=& 2\lambda\, g'' + 2(\alpha\, s_3+\beta\, s_1)e + 2\nu\, s_2 e' + 2(q+\alpha\, s_1)e'' + 2s_3 a, \end{array}\right.$$

respectively. Trying a solution $\bar W$ with

$$\left\{\begin{array}{l} e = \sum_{i=0}^{\infty} e_i\lambda^{-i},\ f = \sum_{i=0}^{\infty} f_i\lambda^{-i},\ g = \sum_{i=0}^{\infty} g_i\lambda^{-i}, \\ e' = \sum_{i=0}^{\infty} e'_i\lambda^{-i},\ f' = \sum_{i=0}^{\infty} f'_i\lambda^{-i},\ g' = \sum_{i=0}^{\infty} g'_i\lambda^{-i}, \\ e'' = \sum_{i=0}^{\infty} e''_i\lambda^{-i},\ f'' = \sum_{i=0}^{\infty} f''_i\lambda^{-i},\ g'' = \sum_{i=0}^{\infty} g''_i\lambda^{-i}, \end{array}\right. \tag{3.55}$$

and taking the initial data

$$e_0 = e'_0 = e''_0 = -1,\ f_0 = g_0 = f'_0 = g'_0 = f''_0 = g''_0 = 0, \tag{3.56}$$

and the integration conditions

$$\left\{\begin{array}{ll} e_i|_{\bar u=0} = f_i|_{\bar u=0} = g_i|_{\bar u=0} = 0, & \\ e'_i|_{\bar u=0} = f'_i|_{\bar u=0} = g'_i|_{\bar u=0} = 0, & i\ge 1, \\ e''_i|_{\bar u=0} = f''_i|_{\bar u=0} = g''_i|_{\bar u=0} = 0, & \end{array}\right. \tag{3.57}$$

then we can have

$$\left\{\begin{array}{rcl} f_{i+1} &=& -\frac{1}{2} f_{i,x} - r_1 a_i - (p+\alpha\, r_1)e_i, \\ g_{i+1} &=& \frac{1}{2} g_{i,x} - s_1 a_i - (q+\alpha\, s_1)e_i, \\ e_{i+1,x} &=& -s_1 b_{i+1} + r_1 c_{i+1} - (q+\alpha\, s_1) f_{i+1} + (p+\alpha\, r_1) g_{i+1}; \end{array}\right. \tag{3.58}$$

$$\left\{\begin{array}{rcl} f'_{i+1} &=& -\frac{1}{2} f'_{i,x} - r_2 a_i - (p+\alpha\, r_1+\mu\, r_2)e'_i - \alpha\, r_2 e_i, \\ g'_{i+1} &=& \frac{1}{2} g'_{i,x} - s_2 a_i - (q+\alpha\, s_1+\mu\, s_2)e'_i - \alpha\, s_2 e_i, \\ e'_{i+1,x} &=& -s_2 b_{i+1} + r_2 c_{i+1} - (q+\alpha\, s_1+\mu\, s_2) f'_{i+1} \\ && +(p+\alpha\, r_1+\mu\, r_2) g'_{i+1} - \alpha\, s_2 f_{i+1} + \alpha\, r_2 g_{i+1}; \end{array}\right. \tag{3.59}$$

and

$$\left\{\begin{array}{rcl} f''_{i+1} &=& -\frac{1}{2} f''_{i,x} - (\alpha\, r_3+\beta\, r_1)e_i - \nu\, r_2 e'_i - (p+\alpha\, r_1)e''_i - r_3 a_i, \\ g''_{i+1} &=& \frac{1}{2} g''_{i,x} - (\alpha\, s_3+\beta\, s_1)e_i - \nu\, s_2 e'_i - (q+\alpha\, s_1)e''_i - s_3 a_i, \\ e''_{i+1,x} &=& -s_3 b_{i+1} + r_3 c_{i+1} - (\beta\, s_1+\alpha\, s_3) f_{i+1} + (\beta\, r_1+\alpha\, r_3) g_{i+1} \\ && -\nu\, s_2 f'_{i+1} + \nu\, r_2 g'_{i+1} - (q+\alpha\, s_1) f''_{i+1} + (p+\alpha\, r_1) g''_{i+1}, \end{array}\right. \tag{3.60}$$

where $i \ge 0$. Under the integration conditions in (3.57), these recursion relations uniquely determine the sequences of $\{e_i, f_i, g_i |\, i\ge 1\}$, $\{e'_i, f'_i, g'_i |\, i\ge 1\}$ and $\{e''_i, f''_i, g''_i |\, i\ge 1\}$.

**Integrable couplings:**

Let us now introduce the enlarged Lax matrices

$$\bar V^{[m]} = M_3(V^{[m]}, V_1^{[m]}, V_2^{[m]}, V_3^{[m]}),\ m\ge 0, \tag{3.61}$$

where $V^{[m]}$ is defined as in (3.24) and

$$V_i^{[m]} = (\lambda^m W_i)_+,\ m \geq 0. \tag{3.62}$$

Then, the enlarged zero curvature equations

$$\bar{U}_{t_m} = \bar{V}_x^{[m]} - [\bar{U}, \bar{V}^{[m]}],\ m \geq 0, \tag{3.63}$$

determine a hierarchy of coupling systems for the AKNS equations in (3.25):

$$\bar{u}_{t_m} = \begin{bmatrix} p_{t_m} \\ q_{t_m} \\ r_{1,t_m} \\ s_{1,t_m} \\ r_{2,t_m} \\ s_{2,t_m} \\ r_{3,t_m} \\ s_{3,t_m} \end{bmatrix} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u,u_1) \\ S_{2,m}(u,u_1,u_2) \\ S_{3,m}(u,u_1,u_2,u_3) \end{bmatrix} = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \\ -2f'_{m+1} \\ 2g'_{m+1} \\ -2f''_{m+1} \\ 2g''_{m+1} \end{bmatrix}, \quad m \geq 0. \tag{3.64}$$

It is direct to check that all members in (3.64) with $m \geq 2$ provide nonlinear tri-integrable couplings for the AKNS equations.

**Hamiltonian structures:**

In order to furnish Hamiltonian structures for the presented tri-integrable couplings in (3.64), we apply the variational identity. Following the general procedure in subsection 2.2, it is direct to compute that

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \alpha\,\eta_2 + \beta\,\eta_4 & \alpha\,\eta_3 & \alpha\,\eta_4 \\ \eta_3 & \alpha\,\eta_3 & \mu\,\eta_3 + \nu\,\eta_4 & 0 \\ \eta_4 & \alpha\,\eta_4 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{3.65}$$

where $\eta_i$, $1 \leq i \leq 4$, are arbitrary constants. Now, the corresponding symmetric and ad-invariant bilinear form on the underlying matrix Lie algebra is given as follows:

$$\begin{aligned} \langle A, B\rangle &= \langle \sigma(A), \sigma(B)\rangle_{\mathbb{R}^{12}} = (a_1, \cdots, a_{12})F(b_1, \cdots, b_{12})^T \\ &= (2a_1b_1 + a_2b_3 + a_3b_2)\eta_1 \\ &\quad +(2a_1b_4 + a_2b_6 + a_3b_5 + 2a_4b_1 + 2\alpha\, a_4b_4 + a_5b_3 + \alpha\, a_5b_6 + a_6b_2 + \alpha\, a_6b_5)\eta_2 \\ &\quad +(2a_1b_7 + a_2b_9 + a_3b_8 + 2\alpha\, a_4b_7 + \alpha\, a_5b_9 + \alpha\, a_6b_8 + 2a_7b_1 + 2\alpha\, a_7b_4 \\ &\quad +2\mu\, a_7b_7 + a_8b_3 + \alpha\, a_8b_6 + \mu\, a_8b_9 + a_9b_2 + \alpha\, a_9b_5 + \mu\, a_9b_8)\eta_3 \\ &\quad +(2a_1b_{10} + a_2b_{12} + a_3b_{11} + 2\alpha\, a_4b_{10} + 2\beta\, a_4b_4 + \beta\, a_5b_6 + \alpha\, a_5b_{12} \\ &\quad +\beta\, a_6b_5 + \alpha\, a_6b_{11} + 2\nu\, a_7b_7 + \nu\, a_8b_9 + \nu\, a_9b_8 + 2\alpha\, a_{10}b_4 \\ &\quad +2a_{10}b_1 + \alpha\, a_{11}b_6 + a_{11}b_3 + \alpha\, a_{12}b_5 + a_{12}b_2)\eta_4, \end{aligned} \tag{3.66}$$

where $A$ and $B$ are defined by

$$A = \sigma^{-1}((a_1, \cdots, a_{12})^T) = M_3(A_1, A_2, A_3, A_4),\ B = \sigma^{-1}((b_1, \cdots, b_{12})^T) = M_3(B_1, B_2, B_3, B_4) \tag{3.67}$$

with $A_i$ and $B_i,\ 1 \leq i \leq 4$, being given by

$$A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix},\ B_i = \begin{bmatrix} b_{3i-2} & b_{3i-1} \\ b_{3i} & -b_{3i-2} \end{bmatrix},\ 1 \leq i \leq 4. \tag{3.68}$$

The bilinear form defined by (3.66) is non-degenerate if and only if the determinant of $F$ is not zero, i.e.,

$$\det(F) = -16{\eta_4}^6 \left(\alpha^2\eta_1 - \alpha\,\eta_2 + \beta\,\eta_4\right)^3 (\mu\,\eta_3 + \nu\,\eta_4)^3 \neq 0. \tag{3.69}$$

Therefore, we can choose values for $\eta_1, \eta_2, \eta_3$ and $\eta_4$ such that $\det(F)$ is non-zero to get non-degenerate bilinear forms over the underling matrix Lie algebra.

It is now direct to compute that

$$\langle \bar{W}, \bar{U}_\lambda\rangle = -2a\,\eta_1 - 2e\,\eta_2 - 2e'\,\eta_3 - 2e''\,\eta_4,$$

and

$$\langle \bar{W}, \bar{U}_{\bar{u}}\rangle = \begin{bmatrix} c\,\eta_1 + g\,\eta_2 + g'\,\eta_3 + g''\,\eta_4 \\ b\,\eta_1 + f\,\eta_2 + f'\,\eta_3 + f''\,\eta_4 \\ \alpha\, g'\,\eta_3 + c\,\eta_2 + (\alpha\,\eta_2 + \beta\,\eta_4)g + \alpha\, g''\,\eta_4 \\ b\,\eta_2 + \alpha\, f'\,\eta_3 + \alpha\, f''\,\eta_4 + (\alpha\,\eta_2 + \beta\,\eta_4)f \\ c\,\eta_3 + (\mu\,\eta_3 + \nu\,\eta_4)g' + \alpha\, g\,\eta_3 \\ b\,\eta_3 + \alpha\,\eta_3 f + (\mu\,\eta_3 + \nu\,\eta_4)f' \\ \alpha\, g\,\eta_4 + c\,\eta_4 \\ b\,\eta_4 + \alpha\, f\,\eta_4 \end{bmatrix}.$$

Thus, the corresponding variational identity with $\gamma = 0$ generates a Hamiltonian structure for the hierarchy of tri-integrable couplings in (3.64):

$$\bar{u}_{t_m} = \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}},\ m\geq 0, \tag{3.70}$$

with the Hamiltonian operator

$$\bar{J} = \left[\begin{array}{cccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \alpha\,\eta_2+\beta\,\eta_4 & \alpha\,\eta_3 & \alpha\,\eta_4 \\ \eta_3 & \alpha\,\eta_3 & \mu\,\eta_3+\nu\,\eta_4 & 0 \\ \eta_4 & \alpha\,\eta_4 & 0 & 0 \end{array}\right]^{-1} \otimes \left[\begin{array}{cc} 0 & -2 \\ 2 & 0 \end{array}\right]. \tag{3.71}$$

and the Hamiltonian functionals

$$\bar{\mathcal{H}}_m = \int \frac{2a_{m+2}\,\eta_1 + 2e_{m+2}\,\eta_2 + 2e'_{m+2}\,\eta_3 + 2e''_{m+2}\,\eta_4}{m+1}\,dx,\ m\geq 0. \tag{3.72}$$

**Recursion operator:**

The recursion relation

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1},\ m\geq 1, \tag{3.73}$$

derived form (3.21), (3.58), (3.59) and (3.60), tells that the recursion operator $\bar{\Phi}$ reads

$$\bar{\Phi} = \bar{\Phi}(\bar{u}) = \left[\begin{array}{cccc} \Phi & 0 & 0 & 0 \\ \Phi_1 & \Phi+\alpha\,\Phi_1 & 0 & 0 \\ \Phi_2 & \alpha\,\Phi_2 & \Phi+\alpha\,\Phi_1+\mu\,\Phi_2 & 0 \\ \Phi_3 & \beta\,\Phi_1+\alpha\,\Phi_3 & \nu\,\Phi_2 & \Phi+\alpha\,\Phi_1 \end{array}\right], \tag{3.74}$$

where $\Phi$ is given as in (3.26) and

$$\Phi_1 = \left[\begin{array}{cc} r_1\partial^{-1}q + (p+\alpha\, r_1)\partial^{-1}s_1 & r_1\partial^{-1}p + (p+\alpha\, r_1)\partial^{-1}r_1 \\ -s_1\partial^{-1}q - (q+\alpha\, s_1)\partial^{-1}s_1 & -s_1\partial^{-1}p - (q+\alpha\, s_1)\partial^{-1}r_1 \end{array}\right],$$

$$\Phi_2 = \left[\begin{array}{cc} (\theta_1+\mu\, r_2)\partial^{-1}s_2 + r_2\partial^{-1}(q+\alpha\, s_1) & (\theta_1+\mu\, r_2)\partial^{-1}r_2 + r_2\partial^{-1}(p+\alpha\, r_1) \\ -(\theta_2+\mu\, s_2)\partial^{-1}s_2 - s_2\partial^{-1}(q+\alpha\, s_1) & -(\theta_2+\mu\, s_2)\partial^{-1}r_2 - s_2\partial^{-1}(p+\alpha\, r_1) \end{array}\right],$$

$$\Phi_3 = \left[\begin{array}{cc} \begin{array}{l}\theta_1\partial^{-1}s_3 + \theta_3\partial^{-1}s_1 \\ +\nu\, r_2\partial^{-1}s_2 + r_3\partial^{-1}q,\end{array} & \begin{array}{l}\theta_1\partial^{-1}r_3 + \theta_3\partial^{-1}r_1 \\ +\nu\, r_2\partial^{-1}r_2 + r_3\partial^{-1}p\end{array} \\ \begin{array}{l}-\theta_2\partial^{-1}s_3 - \theta_4\partial^{-1}s_1 \\ -\nu\, s_2\partial^{-1}s_2 - s_3\partial^{-1}q,\end{array} & \begin{array}{l}-\theta_2\partial^{-1}r_3 - \theta_4\partial^{-1}r_1 \\ -\nu\, s_2\partial^{-1}r_2 - s_3\partial^{-1}p\end{array} \end{array}\right],$$

in which

$$\theta_1 = p+\alpha\, r_1,\ \theta_2 = q+\alpha\, s_1,\ \theta_3 = \alpha\, r_3+\beta\, r_1,\ \theta_4 = \alpha\, s_3+\beta\, s_1.$$

### 3.3.4 Liouville Integrability

It is direct to show by computer algebra systems such as Maple that the enlarged recursion operators $\bar{\Phi}$, defined by (3.49) and (3.74), are all hereditary (Fuchssteiner, 1979), i.e., they satisfy

$$\bar{\Phi}'(\bar{u})[\bar{\Phi}\bar{K}]\bar{S} - \bar{\Phi}\bar{\Phi}'(\bar{u})[\bar{K}]\bar{S} = \bar{\Phi}'(\bar{u})[\bar{\Phi}\bar{S}]\bar{K} - \bar{\Phi}\bar{\Phi}'(\bar{u})[\bar{S}]\bar{K} \tag{3.75}$$

for all enlarged vector fields $\bar{K}$ and $\bar{S}$; and that each pair of $\bar{J}$ and $\bar{M} = \bar{\Phi}\bar{J}$ in the resulting two hierarchies of integrable couplings, (3.45) and (3.70), constitutes a Hamiltonian pair (Magri, 1978), i.e., any linear combination $\bar{N}$ of $\bar{J}$ and $\bar{M}$ satisfies

$$\int (\bar{K})^T\bar{N}'(\bar{u})[\bar{N}\bar{S}]\bar{T}\,dx + \text{cycle}(\bar{K},\bar{S},\bar{T}) = 0 \tag{3.76}$$

for all enlarged vector fields $\bar{K}$, $\bar{S}$ and $\bar{T}$.

Therefore, the bi-integrable couplings in (3.39) and the tri-integrable couplings in (3.64) are bi-Hamiltonian (see, e.g., Magri, 1978; Olver, 1986), and so, all the presented integrable couplings are Liouville integrable.

In particular, we have

$$[\bar{K}_m,\bar{K}_n] = \bar{K}'_m(\bar{u})[\bar{K}_n] - \bar{K}'_n(\bar{u})[\bar{K}_m] = 0,\ m,n\geq 0, \tag{3.77}$$

and

$$\{\bar{\mathcal{H}}_m,\bar{\mathcal{H}}_n\}_{\bar{J}} = \int (\frac{\delta\tilde{H}_m}{\delta\bar{u}})^T\,\bar{J}\frac{\delta\tilde{H}_n}{\delta\bar{u}}\,dx = 0,\ m,n\geq 0. \tag{3.78}$$

These provide infinitely many common commuting symmetries $\{\bar{K}_n|\,n\geq 0\}$ and conserved functionals $\{\bar{\mathcal{H}}_n|\,n\geq 0\}$ .

# 4 SUPER HAMILTONIAN STRUCTURES

If the underlying Lie algebra is superalgebra, we have similar super variational identities. Let $\bar{g}$ be a Lie superalgebra over a supercommutative ring. Then the continuous super variational identity on $\bar{g}$ holds:

$$\frac{\delta}{\delta u}\int \langle W, \frac{\partial U}{\partial \lambda}\rangle \, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle W, \frac{\partial U}{\partial u}\rangle, \tag{4.1}$$

and the discrete super variational identity on $\bar{g}$ holds:

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}}\langle W, \frac{\partial U}{\partial \lambda}\rangle = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle W, \frac{\partial U}{\partial u}\rangle, \tag{4.2}$$

where $\gamma$ is a fixed constant, $U, W \in \bar{g}$ satisfy (2.3) [or (2.4)], i.e., $W_x = [U, W]$ [or $(EW)(EU) = UW$], and $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the Lie superalgebra $\bar{g}$. A special case of the super variational identities are the supertrace identities (Ma et al., 2008):

$$\frac{\delta}{\delta u}\int \mathrm{str}(\mathrm{ad}_W \mathrm{ad}_{\frac{\partial U}{\partial \lambda}})\, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(\mathrm{ad}_W \mathrm{ad}_{\frac{\partial U}{\partial u}}), \tag{4.3}$$

and

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}} \mathrm{str}(\mathrm{ad}_W \mathrm{ad}_{\frac{\partial U}{\partial \lambda}}) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(\mathrm{ad}_W \mathrm{ad}_{\frac{\partial U}{\partial u}}), \tag{4.4}$$

where $\mathrm{ad}_a b = [a, b]$ and str is the supertrace.

If a spectral matrix $U = U(u, \lambda)$ is of order 2, we can make a super generalization to construct a super soliton hierarchy:

$$\bar{U} = U(u,\lambda) + \alpha E_3 + \beta E_4 = \begin{bmatrix} \begin{matrix} & \\ & U(u,\lambda) & \\ & \end{matrix} & \begin{matrix} \alpha \\ \beta \end{matrix} \\ \begin{matrix} \beta & -\alpha \end{matrix} & 0 \end{bmatrix}, \tag{4.5}$$

where $E_3$ and $E_4$ are odd generators of the super sl$(2)$, $u$ is a vector of commuting variables, and $\alpha$ and $\beta$ are anti-commuting variables. Applications of the super-trace identities lead to super integrable systems and super-symmetric integrable systems (see, e.g., Hu, 1997; Ma et al., 2008).

We can also form semi-direct sums of Lie superalgebras and take new enlarged spectral matrices from the resulting semi-direct sums, to construct super integrable couplings. More specifically, we can make $\bar{g} = g \Subset g_c$ with the Lie product:

$$\bar{W} = W + W_c = \left[\bar{U}, \bar{V}\right] = \left[U + U_c, V + V_c\right], \; \bar{U}, \bar{V} \in \bar{g}, \tag{4.6}$$

where

$$W = [U, V] \in g, \; W_c = [U, V_c] + [U_c, V] + [U_c, V_c] \in g_c. \tag{4.7}$$

Applications of the super variational identities will lead to super Hamiltonian structures for super integrable couplings. The procedure for constructing super integrable couplings is almost the same as the one in the classical case. Only one needs to pay particular attention to the anticommuting property of fermionic variables.

One class of the powerful super variational identities are bi-supertrace identities for constructing super Hamiltonian dark equations:

$$\frac{\delta}{\delta u}\int \left[\mathrm{str}\left(W_0\frac{\partial U_1}{\partial \lambda}\right) + \mathrm{str}\left(W_1\frac{\partial U_0}{\partial \lambda}\right)\right] dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left[\mathrm{str}\left(W_0\frac{\partial U_1}{\partial u}\right) + \mathrm{str}\left(W_1\frac{\partial U_0}{\partial u}\right)\right] \tag{4.8}$$

in the continuous case, and

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}}\left[\mathrm{str}\left(W_0\frac{\partial U_1}{\partial \lambda}\right) + \mathrm{str}\left(W_1\frac{\partial U_0}{\partial \lambda}\right)\right] = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left[\mathrm{str}\left(W_0\frac{\partial U_1}{\partial u}\right) + \mathrm{str}\left(W_1\frac{\partial U_0}{\partial u}\right)\right] \tag{4.9}$$

in the discrete case.

# 5 CONCLUDING REMARKS

We proposed a few classes of matrix Lie algebras consisting of block matrices to generate integrable couplings, and successfully constructed dark equations for the KdV equation and the AKNS nonlinear Schrödinger equations, and bi-Hamiltonian bi- and tri-integrable couplings for the AKNS equations. The presented matrix Lie algebras serve as a starting point to construct integrable couplings and the general construction scheme can be applied to the other existing soliton hierarchies like the Dirac hierarchy and the Kaup-Newell hierarchy.

Integrable couplings provide us with valuable insights into the general structure of integrable systems with multi-components. It will be very helpful in building an exhaustive list of integrable systems to collect concrete examples of integrable couplings. The theory of integrable couplings yields diverse hereditary recursion operators in block matrix form, which are difficult to obtain by any direct method. The mathematical theory behind integrable couplings is rich and interesting. We feel that we are only at the beginning of classifying multiple component integrable systems. Exploring specific examples of integrable couplings will help us discover rich mathematical structures that integrable systems possess.

There are many further questions on integrable couplings and their solution theories. We list some of them as follows.

**Super-symmetric zero curvature equations:**

It is an unsolved problem how to generalize zero curvature equations to the super-symmetric case. Even in the special case of $D = 1$ and $N = 1$, it is not clear to us that how one can solve

$$D_x W = [U, W],\ D_x = \partial_\theta + \theta\partial_x,$$

for $W$ such that the super-symmetric zero curvature equation

$$U_t - D_x V + [U, V] = 0$$

generates super-symmetric integrable systems.

**Localness associated with matrix Lie algebras:**

There is a localness problem in generating integrable couplings. For example, based on

$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & \beta A_2 + \alpha A_3 \\ 0 & 0 & A_1 + \alpha A_2 \end{bmatrix},$$

how can one show localness of the sub-systems corresponding to the third block $A_3$ in the resulting integrable couplings? Our examples discussed above showed that the obtained integrable couplings are all local.

**Open question on linear differential equations:**

While solving dark equations, we always need to solve linear PDEs with variable coefficients. This brings us a basic problem about the solution structure of systems of linear ODEs with variable coefficients. How can one represent their general solutions?

Consider a Cauchy problem

$$\dot{x}(t) = A(t)x(t),\ x(0) = x_0 \in \mathbb{R}^n,$$

where $n \geq 2$. It has the solution

$$x(t) = e^{\int_0^t A(s)\,ds} x_0$$

if the coefficient matrix commutes with its integral $[A(t), \int_0^t A(s)\,ds] = 0$. An open question is whether the commutativity condition $[A(t), \int_0^t A(s)\,ds] = 0$ is necessary to guarantee that the vector function $x(t)$ defined above solves the discussed Cauchy problem (Ma et al., 2009).

**Open question on the chain rule:**

A further open question related to the representation of solutions to systems of linear ODEs is on the chain rule of differentiating exponential matrix functions (Ma & Shekhtman, 2010): Is the commutativity condition $[A(t), \int_0^t A(s)\,ds] = 0$ necessary to guarantee the chain rule

$$\frac{d}{dt} e^{\int_0^t A(s)\,ds} = A(t)\, e^{\int_0^t A(s)\,ds}?$$

This chain rule holds if $[A(t), \int_0^t A(s)\,ds] = 0$. Some counterexamples about a weaker question (Ma & Shekhtman, 2010) were presented by using complex matrices satisfying

$$e^C - C - I_n = 0,$$

where $I_n$ is the identity matrix of order $n$.

**Criterion for multivariate polynomials with one zero:**

In generating exact solutions to bilinear differential equations, there is an open question on multivariate polynomials (Ma et al., 2012): How to determine if a real multivariate polynomial has one and only one zero? There are many such polynomials, among which are the following two examples:

$$\begin{aligned} &x^2 + y^2,\ \text{zero } (x, y) = (0, 0); \\ &2x^2 - 6xy + 5y^2 + 2x - 4y + 1,\ \text{zero } (x, y) = (1, 1). \end{aligned}$$

This seems more general than Hilbert's 17th problem, noting that all such multivariate polynomials satisfy the requirement in Hilbert's 17th problem.

There are many other interesting questions on integrable couplings. For instance, what kinds of other non-semisimple matrix Lie algebras can we begin with, to generate bi- or tri-integrable couplings? It is known that Hamiltonian structures exist for the perturbation systems (Ma & Fuchssteiner, 1996; Sakovich, 1998; Ma, 2002; Ma, 2005), but some matrix Lie algebras of enlarged block matrices do not possess any non-degenerate bilinear forms required in the variational identities (Ma, 2003; Ma & Gao, 2009). Are there any concrete criteria which determine if there exist Hamiltonian structures for integrable couplings, even bi- and tri-integrable couplings? How can one compute solution groups for general integrable couplings by symmetry constraints like the perturbation systems (Ma & Zhou, 2001; Ma & Zhou, 2002) or by Darboux transformations engendered through moving frames (Olver, 1999)? A concrete example is the following bi-integrable coupling

$$\begin{cases} u_t = K(u), \\ v_t = K'(u)[v], \\ w_t = K'(u)[w], \end{cases}$$

as we mentioned before. How about its Hamiltonian structure and solution groups by symmetry constraints?

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