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The complex KdV-Burgers equation

Netra Khanal[1]

[1] The University of Tampa, Department of Mathematics
401 W Kennedy Blvd, Tampa, FL 33620, USA
Email: nkhanal@ut.edu.

ABSTRACT: This paper discusses spatially periodic solutions of the complex KdV-Burgers equation. We examine a special series solution of KdV-Burgers equation and prove the convergence and global regularity of such solutions associated with initial data satisfying mild conditions. We also establish the existence and uniqueness of the Fourier series solution with the Fourier modes decaying algebraically in terms of the wave numbers.

Key Words:

Complex KdV-Burgers equation, series solutions, global regularity

1 INTRODUCTION

We consider the complex Kortweg-de Vries(KdV)-Burgers equation

$$u_t - 6uu_x + \alpha u_{xxx} - \nu u_{xx} = 0, \tag{1.1}$$

where $\nu \geq 0$ and $\alpha \geq 0$ are diffusion and dispersion coefficients respectively and $u = u(x,t) = f(x,t) + ig(x,t)$ where f and g denote real and imaginary parts of u. Attention will be paid to the periodic solutions of the equation (1.1) on $x \in \mathbf{T} = [0, 2\pi]$ with the given initial data

$$u(x,0) = u_0(x), \quad x \in \mathbf{T}. \tag{1.2}$$

Recent work by Khanal et al. [6] constructs the finite-time singular solutions of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\, e^{ikx} \tag{1.3}$$

of complex Burgers equation ((1.1) with $\alpha = 0$) corresponding to the initial data $u_0(x) = a\, e^{ix}$. Their work asserts that for any sufficiently large time T, there exists an explicit smooth initial data u_0 such that its corresponding solution blows up at $t = T$. Similar results are also developed in the paper of Poláčik and Šverák [13], in which the complex-valued Burgers equation on the whole line was shown to develop finite-time singularities for compactly supported smooth data. Their proof takes advantage of the explicit solution formula

$$u(x,t) = -3\frac{\int_{-\infty}^{\infty} \frac{x-y}{t} \exp\left[-\frac{|x-y|^2}{2\nu t} - \frac{1}{2\nu}\int_{-\infty}^{y} u_0(s)\, ds\right] dy}{\int_{-\infty}^{\infty} \exp\left[-\frac{|x-y|^2}{2\nu t} - \frac{1}{2\nu}\int_{-\infty}^{y} u_0(s)\, ds\right] dy}$$

obtained via the Hopf-Cole transform. The complex KdV equation

$$u_t - 6uu_x + \alpha u_{xxx} = 0$$

also is known to have several family of solutions that blow up in a finite time ([1],[2],[17]). As one would expect, the behavior of solutions to the complex KdV-Burgers equations is more sophisticated due to the presence of the nonlinearity, dispersion and dissipation.

In this paper, we explore the conditions under which two types of series solutions of (1.1) are global in time. The first kind discussed in [6] assumes the form

$$u(x,t) = \sum_{k=0}^{\infty} a_k(t) e^{ikx} \tag{1.4}$$

and it has the advantage that the coefficients $a_k(t)$'s can be formulated explicitly in terms of $a_k(0)$, α and ν. Consequently, the evolution of $a_k(t)$ can be traced very closely. As discussed in [6], a simple example of the global solutions of (1.1) corresponds to the initial data $u_0(x) = a_0 e^{ix}$ with $|a_0| < 1$ provided ν and α satisfy a suitable condition, say $\nu^2 + 4\alpha^2 \geq 9$ (see Theorem 2.7). For general initial data of the form

$$u_0(x) = \sum_{k=1}^{\infty} a_{0k}\, e^{ikx}$$

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with $|a_{0k}| < 1$, (1.1) possesses a unique local solution (1.3) with $a_k(t)$ given by a finite sum of terms that can be made explicit through an inductive relation. To show the convergence of (1.3) for large time, we estimate $|a_k(t)|$ by counting the total number of terms that it contains. This counting problem is closely related to the number of nonnegative integer solutions to the equation

$$j_1 + 2j_2 + 3j_3 + \cdots + kj_k = k$$

for a fixed integer $k > 0$. Using a result by Hardy and Ramanujan [3], we are able to establish the global regularity of (1.3) under a mild assumption (see Theorem 2.5). In addition, $\|u(\cdot,t)\|_{H^s}$ for any $s \geq 0$ decays exponentially in t for large t.

Inspired by a recent work of Sinai on the Navier-Stokes equations [14] and Khanal et al. on the Kawahara equation [5], we also study the series solution of (1.1) that can be written as

$$u(x,t) = \sum_{k\in \mathbf{Z}\setminus\{0\}} \frac{c(k,t)}{|k|^\gamma}\, e^{ikx} \tag{1.5}$$

where $\gamma > 1$ is a real number and $c(k,t)$ is bounded uniformly in terms of k and t. If $T > 0$ and $R_0 = \sup_{k\in \mathbf{Z}\setminus\{0\}} |c(k,0)|$ satisfy

$$R_0\sqrt{T} \leq C(\gamma)\sqrt{\nu}$$

for some suitable constant $C(\gamma)$, we show that u in (1.5) is a classical solution of (1.1) on $[0,T]$ (Theorems 3.3, 3.4 and 3.5). First, we establish the existence of $c(k,t)$ such that

$$\widehat{u}(k,t) = \frac{c(k,t)}{|k|^\gamma}$$

solves the Fourier transform of the complex KdV-Burgers equation. Second, we verify that u in (1.5) is a weak solution in the distributional sense while the third step proves the bound

$$|c(k,t)| \leq \frac{C}{|k|^{\gamma+l}}$$

where $l > 0$ is any fixed integer. A combination of the last two steps especially implies that u in (1.5) is a classical solution. This process is carried out exactly in the similar fashion as it is done in [5].

The complex KdV, Burgers, and KdV-Burgers equations have recently attracted a good deal of attention. These equations are important both physically and mathematically. Physically these complex equations do arise in the modeling of several physical phenomena ([4],[9],[10]). Mathematically these equations exhibit some remarkable features and are found to be more sophisticated than their real counterpart. For example, the complex KdV is equivalent to a system of two nonlinearly coupled equations and the conservation laws no longer allow the deduction of global bounds and does not lead to the global boundedness of the L^2-norm of its solutions. The study of complex-valued Burgers and KdV-Burgers equations is justified to see the effect of dispersion and dissipation on the solutions of complex KdV. A lot of effort has been devoted recently to the important issue of whether or not their solutions can blow up in a finite time. In 1987, B. Birnir studied the solutions of the complex KdV equation represented by Weierstrass $\mathcal{P}$-function, and proved that they blow up in finite time as a second order pole. In [2], J. Bona and F. Weissler presented some criteria to imply that the solutions of nonlinear, dispersive evolution equations lose regularity in finite time. The papers of Yuan and Wu ([15],[16],[17]) treated the complex KdV and KdV-Burgers equations as systems of two nonlinearly coupled equations and clarified how the potential singularities of the real part are related to those of the imaginary part. Very recently Y. Li [12] obtained simple explicit formulas for finite-time blowup solutions of the complex KdV equation through Darboux transform. Another example showing the differences between the real-valued and complex-valued solutions is the Navier-Stokes equations. Li and Sinai [11] recently showed that the complex solutions of the 3D Navier-Stokes equations corresponding to large parameter family of initial data blow up in finite time but it remains open whether or not classical solutions of the 3D incompressible Navier-Stokes equations can develop finite-time singularities.

We organize the rest of this paper as follows. The second section focuses on the series solutions of the form (1.4) of complex-valued Kdv-Burgers equation and presents the summary of Theorems established in [6]. The third section is devoted to the series solution of the form (1.5) and details the global regularity results(see Theorems 3.2.3.3,3.4).

2 A special series solution

This section seeks solutions of the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) e^{ikx}. \tag{2.6}$$

to the initial-value problem for the complex KdV-Burgers equation

$$\begin{cases} u_t - 6uu_x + \alpha u_{xxx} - \nu u_{xx} = 0, & x \in \mathbf{T},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{T} \end{cases} \tag{2.7}$$

where $\alpha \geq 0$ and $\nu \geq 0$.

Local existence and uniqueness result on solutions of the form (2.6) to the complex-valued KdV-Burgers type equation

$$u_t - 6uu_x + \nu(-\Delta)^\gamma u + \alpha, u_{xxx} = 0, \tag{2.8}$$

has been discussed in detail in [6]. The equation (2.8) reduces to the complex KdV-Burgers equation when $\gamma = 1$. The fractal Laplacian $(-\Delta)^\gamma$ is defined through Fourier transform,

$$\widehat{(-\Delta)^\gamma} u(\xi) = |\xi|^{2\gamma}\, \widehat{u}(\xi).$$

The other result proved in [6] asserts that if the L^2-norm of a solution of (2.8) is bounded on $[0,T]$, then all higher derivatives are bounded and no singularity is possible on $[0,T]$. We first present two major results on local well-posedness without proofs.

Theorem 2.1. *Consider (2.8) with $\gamma > \frac{1}{2}$. Let $s > \frac{1}{2}$. Assume $u_0 \in H^s(\mathbf{T})$ has the form*

$$u_0(x) = \sum_{k=1}^{\infty} a_{0k}\, e^{ikx}. \tag{2.9}$$

Then there exists $T = T(\|u_0\|_{H^s})$ such that (2.8) with the initial data u_0 has a unique solution $u \in C([0,T); H^s) \cap L^2([0,T); \mathring{H}^{s+\gamma})$ that assumes the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\, e^{ikx}.$$

In the case when $\gamma \geq 1$, we can actually show that no finite-time singularity is possible if we know that the L^2-norm is bounded *a priori*. In fact, the following theorem states that the L^2-norm controls all higher-order derivatives.

Theorem 2.2. *Let $T > 0$ and let u be a weak solution of (2.8) with $\gamma \geq 1$ on the time interval $[0,T]$. If we know a priori that $u \in L^\infty([0,T]; L^2) \cap L^2([0,T]; \mathring{H}^\gamma)$, namely*

$$M_0 \equiv \sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^2}^2 + \nu \int_0^T \|\Lambda^\gamma u(\cdot,t)\|_{L^2}^2\, dt < \infty, \tag{2.10}$$

then, for any integer $k > 0$,

$$M_k \equiv \sup_{t\in[0,T]} \|u^{(k)}(\cdot,t)\|_{L^2}^2 + \nu \int_0^T \|\Lambda^{k+\gamma} u(\cdot,t)\|_{L^2}^2\, dt < \infty.$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and $u^{(k)}$ denotes any partial derivative of order k.

Now, we study the global regularity of solutions of the form (2.6) for complex KdV-Burgers equation (2.7). Here, two major results are established. Theorem 2.5 presents a general conditional global regularity result and Theorem 2.7 asserts the global regularity of (2.6) for a special case. Assume the initial data u_0 is of the form

$$u_0(x) = \sum_{k=1}^{\infty} a_{0k}\, e^{ikx} \tag{2.11}$$

and is in H^s with $s > \frac{1}{2}$. According to Theorem 2.1, (2.7) has a unique local solution $u \in C([0,T); H^s)$ of the form (2.6) for some $T > 0$. To study the global regularity of (2.6), we explore the structure of $a_k(t)$ and obtain the following two propositions.

Proposition 2.3. *If (2.6) solves (2.7), then $a_k(t)$ can be written as*

$$a_k(t) = \sum_{k\le h\le k^2,\, k\le l\le k^3} a_{k,\,h,\,l}\, e^{-(\nu h - \alpha i l)t} \tag{2.12}$$

where $a_{k,\,h,\,l}$ consists of a finite number of terms of the form

$$C(\alpha,\nu,k,h,l,j_1,\cdots,j_k)\, a_{01}^{j_1}\, a_{02}^{j_2} \cdots a_{0k}^{j_k} \tag{2.13}$$

with j_1, j_2 ,$\cdots$, j_k being nonnegative integers and satisfying

$$j_1 + 2j_2 + \cdots + kj_k = k. \tag{2.14}$$

Proposition 2.4. *Let $k \geq 1$ be an integer. Let $U(k) = k^2 - 2k + 2$ and $V(k) = k^3 - 3k^2 + 3k$. The coefficients $a_{k,h,l}$ in (2.12) have the following properties*

(1) For $k \le h < k^2$ *and* $k \le l < k^3$,

$$a_{k,h,l} = \frac{3ik}{\nu(k^2-h) - i\alpha(k^3-l)} \sum_{k_1+k_2=k} \sum_{h_1+h_2=h} \sum_{1_1+1_2=l} \alpha_{k_1,h_1,l_1} \alpha_{k_2,h_2,l_2} \tag{2.15}$$

(2) For $h = k^2$ *and* $l = k^3$,

$$a_{k,k^2,k^3} = a_k(0) - \sum_{k \le h < k^2} \sum_{k \le l < k^3} a_{k,h,l} \tag{2.16}$$

(3) For $U(k) < h < k^2$ *or* $V(k) < l < k^3$,

$$a_{k,h,l} = 0. \tag{2.17}$$

Proof of Proposition 2.3. If (2.6) solves (2.7) , then $a_k(t)$ solves the ordinary differential equation

$$\frac{d}{dt} a_k(t) + (\nu k^2 - \alpha i k^3) a_k(t) - 3ik \sum_{k_1+k_2=k} a_{k_1}(t)\, a_{k_2}(t) = 0.$$

The equivalent integral form is given by

$$a_k(t) = e^{-(\nu k^2 - \alpha i k^3)t} \left[a_{0k} + 3ik \int_0^t e^{(\nu k^2 - \alpha i k^3)\tau} \sum_{k_1+k_2=k} a_{k_1}(\tau)\, a_{k_2}(\tau)\, d\tau \right]. \tag{2.18}$$

It is easy to show through an inductive process that a_k is of the form (2.12). In addition, for $k \le h < k^2$ and $k \le l < k^3$, the term in (2.13) with fixed $j_1, j_2, \cdots, j_k$ satisfying

$$j_1 + 2j_2 + \cdots + kj_k = k$$

can be expressed as

$$\begin{aligned} &C(\alpha,\nu,k,h,l,j_1,\cdots,j_k)\, a_{01}^{j_1} a_{02}^{j_2} \cdots a_{0k}^{j_k} \\ &= \frac{3ik}{\nu(k^2-h) - i\alpha(k^3-l)} \sum_{m_1+n_1=j_1} \cdots \sum_{m_k+n_k=j_k} C(\alpha,\nu,k_1,h_1,l_1,m_1,\cdots,m_{k_1}) \\ &\quad \times C(\alpha,\nu,k_2,h_2,l_2,n_1,\cdots,n_{k_2})\, a_{01}^{m_1+n_1} a_{02}^{m_2+n_2} \cdots a_{0k}^{m_k+n_k} \end{aligned} \tag{2.19}$$

where the indices satisfy

$$\begin{aligned} &1 \le k_1 \le k, \quad 1 \le k_2 \le k, \quad k_1 + k_2 = k, \\ &k_1 \le h_1 \le k_1^2, \quad k_2 \le h_2 \le k_2^2, \quad h_1 + h_2 = h, \\ &k_1 \le l_1 \le k_1^3, \quad k_2 \le l_2 \le k_2^3, \quad l_1 + l_2 = l, \\ &m_1 + n_1 = j_1, \quad m_2 + n_2 = j_2, \quad \cdots, \quad m_k + n_k = j_k. \\ &(m_r = 0 \quad \text{for } r > k_1 \text{ and } \quad n_r = 0 \quad \text{for } r > k_2) \\ &m_1 + 2m_2 + \cdots + k_1 m_{k_1} = k_1, \quad n_1 + 2n_2 + \cdots + k_2 n_{k_2} = k_2. \end{aligned}$$

When $h = k^2$ and $l = k^3$,

$$C(\alpha,\nu,k,k^2,k^3,j_1,j_2,\cdots,j_k) = \begin{cases} 1 \quad \text{for} \quad (j_1,j_2,\cdots,j_k) = (0,0,\cdots,1), \\ -C(\alpha,\nu,k,h,l,j_1,j_2,\cdots,j_k) \quad \text{otherwise} \end{cases} \tag{2.20}$$

for some $h < k^2$ and $l < k^3$. To illustrate these formulas, we list a_k for $k = 1, 2, 3$,

$$\begin{aligned} a_1(t) &= a_{01}\, e^{-(\nu - i\alpha)t}, \\ a_2(t) &= \frac{6i}{2\nu - 6\alpha i} a_{01}^2\, e^{-(2\nu - 2\alpha i)t} + \left[a_{02} - \frac{6i}{2\nu - 6\alpha i} a_{01}^2 \right] e^{-(4\nu - 8i\alpha)t}, \\ a_3(t) &= \frac{108 a_{01}^3}{(2\nu - 6\alpha i)(6\nu - 24\alpha i)} e^{(-3\nu + 3\alpha i)t} \\ &\quad + \left[\frac{18i a_{01} a_{02}}{4\nu - 18\alpha i} - \frac{108 a_{01}^3}{(2\nu - 6\alpha i)(4\nu - 18\alpha i)} \right] e^{(-5\nu + 9i\alpha)t} \\ &\quad + \left[a_{03} - \frac{18i a_{01} a_{02}}{4\nu - 18\alpha i} + \frac{108 a_{01}^3}{(2\nu - 6\alpha i)(4\nu - 18\alpha i)} - \frac{108 a_{01}^3}{(2\nu - 6\alpha i)(6\nu - 24\alpha i)} \right] \\ &\qquad \times e^{(-9\nu + 27\alpha i)t}. \end{aligned}$$

Proof of Proposition 2.4. (2.15) follows from a simple induction. (2.16) is obtained by set $t = 0$ in (2.12). To show (2.17), we notice that the second summation in (2.15) is over $h_1 + h_2 = h$ with $k_1 \le h_1 \le k_1^2$ and $k_2 \le h_2 \le k_2^2$ while the third summation is over $l_1 + l_2 = l$ with $k_1 \le l_1 \le k_1^3$ and $k_2 \le l_2 \le k_2^3$. Thus,

$$\begin{aligned} h &= h_1 + h_2 \le k_1^2 + k_2^2 = k^2 - 2k_1 k_2 \le k^2 - 2(k-1) = U(k), \\ l &= l_1 + l_2 \le k_1^3 + k_2^3 = k^3 - 3k\, k_1 k_2 \le k^3 - 3k(k-1) = V(k). \end{aligned}$$

That means, $a_{k,h,l} = 0$ if $U(k) < h < k^2$ and $V(k) < l < k^3$.

Theorem 2.5. *Consider (2.7) with $\nu > 0$. Assume $u_0 \in H^s(\mathbf{T})$ with $s > \frac{1}{2}$ can be represented in the form (2.11) with*

$$|a_{0k}| \le 1, \quad k = 1, 2, \cdots \tag{2.21}$$

If we have the uniform bound

$$|C(\alpha, \nu, k, h, l, j_1, \cdots, j_k)| \le C_0(\alpha, \nu) \tag{2.22}$$

for all $k \ge 1$, $k \le h < k^2$, $k \le l < k^3$ and $(j_1, j_2, \cdots, j_k)$ satisfying (2.14), then (2.7) has a unique global solution u given by (2.6). In addition, for any $s \ge 0$, there are $T_0 > 0$ and $\delta > 0$ such that for any $t \ge T_0$,

$$\|u(\cdot, t)\|_{H^s} < \frac{C(\alpha, \nu, s)}{1 - e^{-\nu t}}\, e^{-\delta \nu k t} \tag{2.23}$$

where C is a constant depending on α, ν and s only.

We remark that the assumption in (2.22) can be verified for the case when $a_{01} > 0$ and $a_{02} = a_{03} = \cdots = 0$. We assume that ν and α satisfy $\nu^2 + 9\alpha^2 \ge 36$ and show by induction that

$$|C(\alpha, \nu, k, h, l, j_1, \cdots, j_k)| \le 1.$$

Since $a_{02} = a_{03} = \cdots = 0$, these coefficients are nonzero only if $j_1 = k$ and $j_2 = j_3 = \cdots = j_k = 0$. For any $k \le h < k^2$ and $k \le l < k^3$, we have, according to (2.19),

$$\begin{aligned}
&|C(\alpha, \nu, k, h, l, j_1, \cdots, j_k)| \\
&\quad \le \left| \frac{3ik}{\nu(k^2 - h) - i\alpha(k^3 - l)} \right| \sum_{m_1 + n_1 = j_1} |C(\alpha, \nu, k_1, h_1, l_1, m_1, \cdots, m_{k_1})| \\
&\qquad \times |C(\alpha, \nu, k_2, h_2, l_2, n_1, \cdots, n_{k_2})|.
\end{aligned}$$

For $j_1 = k$, the number of terms in the summation $m_1 + n_1 = j_1$ is at most k. By the inductive assumption,

$$|C(\alpha, \nu, k, h, l, j_1, \cdots, j_k)| \le \frac{3k^2}{\sqrt{\nu^2(k^2 - h)^2 + \alpha^2(k^3 - l)^2}}$$

Applying (2.17), $h \le U(k) \equiv k^2 - 2k + 2$ and $l \le V(k) \equiv k^3 - 3k^2 + 3k$ and thus $|C(\alpha, \nu, k, h, l, j_1, \cdots, j_k)| \le 1$ by taking into account the assumption on ν and α. When $h = k^2$ and $l = k^3$, the boundedness of the coefficient follows from (2.20).

The proof of Theorem 2.5 involves a very classical problem in number theory, namely the number of integer solutions $(j_1, j_2, \cdots, j_k)$ to the equation defined in (2.14) for a given positive integer k. This problem is not as simple as it may look like. An upper bound and an asymptotic approximation for the number of nonnegative solutions are given by G.H. Hardy and S. Ramanujan [3], as stated in the following lemma.

Lemma 2.6. *Let $k > 0$ be an integer and let N_k denote the number of nonnegative solutions to the equation*

$$j_1 + 2j_2 + \cdots + kj_k = k.$$

Then, for some constant C_1,

$$N_k < \frac{C_1}{k}\, e^{2\sqrt{2\,k}}.$$

In addition, N_k has the following asymptotic behavior:

$$N_k \quad \sim \quad \frac{1}{4\sqrt{3}k} e^{\pi\sqrt{\frac{2k}{3}}}, \quad \text{as } k \to \infty.$$

Proof of Theorem 2.5. Applying (2.21) and (2.22), we obtain the following bound for $a_{k,\,h,\,l}$ in (2.12)

$$|a_{k,\,h,\,l}| \le C_0(\alpha, \nu)\, N_k \le \frac{C_2}{k}\, e^{2\sqrt{2\,k}},$$

where $C_2 = C_0 C_1$ and we have used Lemma 2.6. Therefore,

$$\begin{aligned}
|a_k(t)| \quad &\le \quad \sum_{k \le h \le k^2} \sum_{k \le l \le k^3} |a_{k,h,l}|\, e^{-\nu h t} \\
&\le \quad C_2\, (k^2 - 1)\, e^{2\sqrt{2}\,\sqrt{k}}\, \frac{e^{-\nu k t}}{1 - e^{-\nu t}}.
\end{aligned} \tag{2.24}$$

For any fixed $t > 0$, we can choose $K = K(\nu)$ and $0 < M = M(\nu) < 1$ such that

$$|a_k(t)| \le \frac{C_2}{1 - e^{-\nu t}}\, M^k \quad \text{for } k \ge K.$$

Therefore, u represented by (2.6) converges for any $t > 0$. In addition, $u(\cdot, t) \in H^s$ for any $s \ge 0$. To see the exponential decay of $\|u(\cdot, t)\|_{H^s}$ for large time, we choose $T_0 = T_0(\nu, s)$ such that for any $t \ge T_0$ and $k \ge 1$

$$(1 + k^2)^s |a_k(t)|^2 \le C_2\, M_1^k\, \frac{e^{-\delta\, \nu k t}}{1 - e^{-\nu t}},$$

where $M_1 > 0$ and $\delta > 0$ are some constants. This bound then implies (2.23). This completes the proof of Theorem 2.5.

We finally present a direct proof of the fact that (2.6) is global in time for special case $a_{02} = a_{03} = \cdots = 0$.

Theorem 2.7. *Consider (2.7) with ν and α satisfying $\nu^2 + 4\alpha^2 \geq 9$. If*

$$u_0(x) = a_{01}\, e^{ix} \quad \textit{with} \quad |a_{01}| < 1,$$

then (2.7) has a unique global solution, which can be represented by (2.6). In addition, for any $s \geq 0$, $u(\cdot, t) \in H^s$ for all $t \geq 0$.

Proof. We prove by induction that, for any $t > 0$,

$$|a_k(t)| \leq |a_{01}|^k, \quad k = 1, 2, \cdots . \tag{2.25}$$

Obviously, $|a_1(t)| \leq |a_{01}|$. To prove (2.25) for $k \geq 2$, we recall (2.18), namely

$$a_k(t) = 3ik\, e^{-(\nu k^2 - \alpha i k^3)t} \int_0^t e^{(\nu k^2 - \alpha i k^3)\tau} \sum_{k_1 + k_2 = k} a_{k_1}(\tau)\, a_{k_2}(\tau)\, d\tau.$$

Since $\nu^2 + 4\alpha^2 \geq 9$, we have

$$|a_2(t)| \leq \left| \frac{3}{2\nu - 4\alpha i} \right| |a_{01}|^2 \left(1 - e^{-(4\nu - 8\alpha i)t}\right) \leq |a_{01}|^2$$

and more generally,

$$|a_k(t)| \leq \left| \frac{3(k-1)}{\nu\, k - \alpha i\, k^2} \right| |a_{01}|^k \left(1 - e^{-(\nu\, k^2 - \alpha\, i\, k^3)t}\right) \leq |a_{01}|^k.$$

It is then clear that (2.6) converges in H^s with $s \geq 0$ for any $t \geq 0$. This completes the proof of Theorem 2.7.

3 Fourier series solution

This section is devoted to full series solutions to the initial-value problem for the complex KdV-Burgers equation (2.7). Assume the initial data u_0 is of the form

$$u_0(x) = \sum_{k \neq 0} \frac{c_0(k)}{|k|^\gamma}\, e^{ikx} \tag{3.26}$$

and write its corresponding solution $u = u(x, t)$ as the series

$$u(x, t) = \sum_{k \neq 0} \widehat{u}(k, t)\, e^{ikx}.$$

Then the coefficient $\widehat{u}(k, t)$ satisfies

$$\widehat{u}(k, t) = e^{(-\nu k^2 + i\alpha k^3)t} \widehat{u}_0(k) + 3ik \int_0^t e^{(-\nu k^2 + i\alpha k^3)(t-s)} \sum_{j \neq 0, j \neq k} \widehat{u}(j, s)\, \widehat{u}(k - j, s)\, ds$$

and, if $\widehat{u}(k, t) = \frac{c(k,t)}{|k|^\gamma}$, then

$$\begin{aligned} c(k, t) \;\; &= \;\; e^{(-\nu k^2 + i\alpha k^3)t} c_0(k) \\ &\quad + 3i\, k|k|^\gamma \int_0^t e^{(-\nu k^2 + i\alpha k^3)(t-s)} \sum_{j \neq 0, j \neq k} \frac{c(j, s)}{|j|^\gamma}\, \frac{c(k - j, s)}{|k - j|^\gamma}\, ds. \end{aligned} \tag{3.27}$$

The goal here is to rigorously establish the existence and uniqueness of such solutions and to understand if they solve (2.7) in the classical sense. We now consider the functional framework discussed in [5]. Here, $X_{\gamma,T}$ denotes the functional space of periodic functions $g = g(x, t)$ on $\mathbf{T} \times [0, T]$ whose fourier coefficient $\widehat{g}(k, t)$ satisfies

$$\widehat{g}(k, t) = \frac{c(k, t)}{|k|^\gamma} \quad \text{for } k \in \mathbf{Z} \setminus \{0\}$$

with

$$\|c\| \equiv \sup_{0 \leq t \leq T} \sup_{k \in \mathbf{Z} \setminus \{0\}} |c(k, t)| < \infty.$$

Definition 3.1. *Let $\gamma > 1$ and $T > 0$. Assume $u \in X_{\gamma,T}$ has the form*

$$u(x, t) = \sum_{k \neq 0} \widehat{u}(k, t)\, e^{ikx} \quad \textit{with} \quad \widehat{u}(k, t) = \frac{c(k, t)}{|k|^\gamma}.$$

Then u is called a series solution of (2.7) if $c(k, 0) = c_0(k)$ and $c(k, t)$ satisfies (3.27) for $t \in [0, T]$.

Proofs of theorems given in this section are similar to the ones described in [5]. The theory discussed in [5] is related to the complex Kawahara equation with dissipation but our work here is related to the complex valued KdV-Burgers equation. Before stating these theorems, we start with a lemma.

Lemma 3.2. *For any $\gamma > 1$ and any integer $k \neq 0$,*

$$\sum_{j\neq 0, j\neq k} \frac{1}{|j|^\gamma |k-j|^\gamma} \leq \frac{C(\gamma)}{|k|^\gamma},$$

where $C(\gamma)$ is a constant independent of k.

The detail of the proof of this lemma is given in [5]. The following theorem establishes the existence and uniqueness of solutions defined in 3.1.

Theorem 3.3. *Consider the initial-value problem for the complex KdV-Burgers equation (2.7). Let $\gamma > 1$ and assume $u_0 \in X_\gamma$ has the form (3.26). If $R_0 \equiv \|u_0\|_{X_\gamma}$ and $T > 0$ satisfy*

$$C(\gamma)\sqrt{T}\, R_0 < \sqrt{\nu}$$

for some suitable constant $C = C(\gamma)$, then (2.7) has a unique series solution $u \in X_{\gamma,T}$. In addition,

$$\|u\|_{X_{\gamma,T}} < 2R_0.$$

Proof of Theorem 3.3. The approach is the method of successive approximation. For each $k \in \mathbf{Z}\setminus\{0\}$, define for $n = 1, 2, \cdots$,

$$\begin{aligned} c^{(0)}(k,t) &= e^{(-\nu k^2 + i\alpha k^3)t} c_0(k), \\ c^{(n)}(k,t) &= e^{(-\nu k^2 + i\alpha k^3)t} c_0(k) \\ &\quad + 3i\, k|k|^\gamma \int_0^t e^{(-\nu k^2 + i\alpha k^3)(t-s)} \sum_{j\neq 0, j\neq k} \frac{c^{(n-1)}(j,s)}{|j|^\gamma} \frac{c^{(n-1)}(k-j,s)}{|k-j|^\gamma}\, ds. \end{aligned}$$

It suffices to show, for some $\theta \in (0,1)$,

$$\|c^{(n)}\| \leq 2R_0, \tag{3.28}$$

$$\|c^{(n)} - c^{(n-1)}\| \leq \theta \|c^{(n-1)} - c^{(n-2)}\|. \tag{3.29}$$

We prove (3.28) by induction. Assume (3.28) holds for all $n \leq m$. Then

$$|c^{(m+1)}(k,t)| \leq e^{-\nu k^2 t} R_0 + \frac{C}{\nu} |k|^{\gamma-1} (1 - e^{-\nu |k|^2 t}) \|c^{(m)}\|^2 \sum_{j\neq 0, j\neq k} \frac{1}{|j|^\gamma |k-j|^\gamma}.$$

Applying Lemma 3.2 and the inductive assumption, we have

$$|c^{(m+1)}(k,t)| \leq e^{-\nu k^2 t} R_0 + \frac{C(\gamma)}{\nu} |k|^{-1} (1 - e^{-\nu|k|^2 t}) R_0^2. \tag{3.30}$$

It is easily verified that, for any $k \neq 0$ and $t \geq 0$,

$$|k|^{-1}(1 - e^{-\nu|k|^2 t}) \leq (\nu\, t)^{\frac{1}{2}}.$$

Consequently,

$$\|c^{(m+1)}\| \leq R_0 + \frac{C(\gamma)}{\nu^{\frac{1}{2}}} T^{\frac{1}{2}} R_0^2.$$

If

$$T^{\frac{1}{2}} R_0 < \frac{\nu^{\frac{1}{2}}}{C(\gamma)}, \tag{3.31}$$

then

$$\|c^{(m+1)}\| \leq 2R_0.$$

To prove (3.29), consider the difference

$$\begin{aligned} |c^{(n)}(k,t) - c^{(n-1)}(k,t)| &= 3|k|^{\gamma+1} e^{-\nu k^2 t} \\ &\quad \times \int_0^t e^{\nu k^2 s} \sum_{j\neq 0, j\neq k} \frac{|c^{(n-1)}(j,s) c^{(n-1)}(k-j,s) - c^{(n-2)}(j,s) c^{(n-2)}(k-j,s)|}{|j|^\gamma |k-j|^\gamma}\, ds. \end{aligned}$$

Writing

$$\begin{aligned} c^{(n-1)}(j,s)\, c^{(n-1)}(k-j,s) &- c^{(n-2)}(j,s)\, c^{(n-2)}(k-j,s) \\ &= [c^{(n-1)}(j,s) - c^{(n-2)}(j,s)]\, c^{(n-1)}(k-j,s) \\ &\quad + c^{(n-2)}(j,s)\, [c^{(n-1)}(k-j,s) - c^{(n-2)}(k-j,s)] \end{aligned}$$

and estimating as in the proof of (3.28), we obtain

$$|c^{(n)}(k,t) - c^{(n-1)}(k,t)| \le \frac{C(\gamma)}{\nu^{\frac{1}{2}}}\, t^{\frac{1}{2}} R_0 \|c^{(n-1)} - c^{(n-2)}\|.$$

When (3.31) is satisfied, then

$$\|c^{(n)} - c^{(n-1)}\| \le \theta \|c^{(n-1)} - c^{(n-2)}\|$$

with

$$\theta = \frac{C(\gamma)}{\nu^{\frac{1}{2}}}\, T^{\frac{1}{2}} R_0 < 1.$$

(3.28) and (3.29) allow us to construct the limit of $c^{(n)}(k,t)$ as

$$c(k,t) = c^{(1)}(k,t) + \sum_{n=1}^{\infty} \left(c^{(n+1)}(k,t) - c^{(n)}(k,t) \right).$$

Going through a simple limiting process, we can show that $c(k,t)$ satisfies (3.27). In addition, by letting $m \to \infty$ in (3.30), we have

$$|c(k,t)| \le e^{-\nu k^2 t} R_0 + \frac{C(\gamma)}{\nu} |k|^{-1} R_0^2, \tag{3.32}$$

which forms the basis for further regularity estimates. This completes the proof of Theorem 3.3.

We now prove that for any $t > 0$, the series solution $u = u(x,t)$ in Theorem 3.3 is actually a classical solution. We divide this process into two steps. First, we show it is a weak solution in the standard distributional sense.

Theorem 3.4. *Assume the conditions of Theorem 3.3 and let u be the series solution obtained there. Then u is a weak solution in the sense that*

$$\int_0^T \int_{\mathbf{T}} u\,(\phi_t - 6u\phi_x + \alpha\phi_{xxx} - \nu\phi_{xx})\, dxdt - \int_{\mathbf{T}} u_0(x)\phi(x,0)\, dx = 0$$

for any $\phi \in C_0^\infty(\mathbf{T} \times [0,T))$.

Proof of Theorem 3.4. Recall that

$$u(x,t) = \sum_{k\neq 0} \widehat{u}(k,t) e^{ikx} \quad \text{with} \quad \widehat{u}(k,t) = \frac{c(k,t)}{|k|^\gamma}.$$

Let $N > 0$ be an integer. Consider

$$u_N(x,t) = \sum_{|k|\le N, k\neq 0} \widehat{u}(k,t)\, e^{ikx}.$$

To derive the equation for u_N, we multiply (3.27) by $\frac{1}{|k|^\gamma}$ and differentiate with respect to t to get

$$\frac{d}{dt}\widehat{u}(k,t) = (-\nu k^2 + i\alpha k^3)\, \widehat{u}(k,t) + 3i\, k \sum_{j\neq 0, j\neq k} \widehat{u}(j,t)\, \widehat{u}(k-j,t).$$

Multiplying this equation by e^{ikx} and summing over $|k| \le N$ ($k \neq 0$), we have

$$\partial_t u_N - 6u_N\, (u_N)_x + \alpha (u_N)_{xxx} - \nu (u_N)_{xx} = R_N, \tag{3.33}$$

where R_N is given by

$$R(x,t) = 3 \sum_{|k|\le N, k\neq 0} (ik) e^{ikx} \sum_{|j|>N} \widehat{u}(j,t)\, \widehat{u}(k-j,t).$$

Multiplying (3.33) by $\phi \in C_0^\infty(\mathbf{T} \times [0,T))$ yields

$$\int_0^T \int_{\mathbf{T}} u_N\, (\phi_t - 6u_N\phi_x + \alpha\phi_{xxx} - \nu\phi_{xx})\, dxdt - \int_{\mathbf{T}} u_0(x)\phi(x,0)\, dx = \int_0^T \int_{\mathbf{T}} R_N \phi\, dxdt.$$

Since $u_N(\cdot,t) \to u(\cdot,t)$ in L^2 uniformly for $t \in [0,T]$, we obtain by letting $N \to \infty$

$$\int_0^T \int_{\mathbf{T}} u\, (\phi_t - 6u\phi_x + \alpha\phi_{xxx} - \nu\phi_{xx})\, dxdt - \int_{\mathbf{T}} u_0(x)\phi(x,0)\, dx = \lim_{N\to\infty} \int_0^T \int_{\mathbf{T}} R_N \phi\, dxdt. \tag{3.34}$$

To show the limit on the right is zero, we use the basic inequality

$$\int_{\mathbf{T}} R_N \phi \, dx \le \left(\int_{\mathbf{T}} R_N^2(x,t)\, dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{T}} \phi(x,t)\, dx \right)^{\frac{1}{2}}$$

and show $\|R_N\|_{L^2} \to 0$. This can be done as follows. Because of the bound

$$\int_{\mathbf{T}} R_N^2(x,t)\, dx \le C \left[\sum_{k=1}^{N} k^2 \sum_{|j|>N} \frac{1}{|j|^{2\gamma}|j-k|^{2\gamma}} + \sum_{k=-N}^{-1} k^2 \sum_{|j|>N} \frac{1}{|j|^{2\gamma}|j-k|^{2\gamma}} \right]$$

for some constant C, it suffices to consider

$$\sum_{k=1}^{N} k^2 \sum_{j>N} \frac{1}{|j|^{2\gamma}|j-k|^{2\gamma}} \le \frac{1}{(N+1)^{2\gamma}} \sum_{k=1}^{N} \frac{k^2}{(N-k+1)^{2\gamma}} \sum_{j\ge N+1} \frac{1}{(1+\frac{j-N-1}{N-k+1})^{2\gamma}}.$$

The last summation can be bounded by an integral,

$$\sum_{j\ge N+1} \frac{1}{(1+\frac{j-N-1}{N-k+1})^{2\gamma}} \le 1 + \int_0^\infty \frac{1}{(1+\frac{x}{N-k+1})^{2\gamma}}\, dx = 1 + \frac{N-k+1}{2\gamma-1}.$$

Thus,

$$\sum_{k=1}^{N} k^2 \sum_{j>N} \frac{1}{|j|^{2\gamma}|j-k|^{2\gamma}} \le \frac{C(\gamma)}{(N+1)^{2\gamma}} \sum_{k=1}^{N} \frac{k^2}{(N-k+1)^{2\gamma-1}} \le \frac{C(\gamma)}{N^{4\gamma-4}}.$$

For $\gamma > 1$, it approaches zero as $N \to \infty$. It then follows from (3.34) that u satisfies the weak formulation. This completes the proof of Theorem 3.4.

The following theorem asserts the regularity of u.

Theorem 3.5. *Assume the conditions of Theorem 3.3 and let u be the series solution obtained there. Then, for any $t_0 > 0$ and nonnegative integer m,*

$$u \in C^1([t_0,T); H^m). \tag{3.35}$$

In particular, this regularity result with Theorem 3.4 implies that u is a classical solution of the complex KdV-Burgers equation (2.7).

Proof of Theorem 3.5. Obviously $u \in L^2([0,T); L^2)$. Fix $t \in (0,T)$. Inserting the simple inequality

$$e^{-\nu k^2 t} \le \frac{1}{|k|} e^{-\nu t} \quad \text{for any } k \in \mathbf{Z} \setminus \{0\}$$

in (3.32), we find

$$c(k,t) = \frac{\widetilde{c}(k,t)}{|k|} \tag{3.36}$$

with

$$|\widetilde{c}(k,t)| \le \widetilde{R}_0 \quad \text{for all } k \neq 0 \text{ and } 0 < t < T.$$

Then $u(x,t)$ can be represented as

$$u(x,t) = \sum_{k\neq 0} \frac{\widetilde{c}(k,t)}{|k|^{\gamma+1}} e^{ikx}.$$

In particular, $u(\cdot,t) \in H^1(\mathbf{T})$. An iterative process would allow us to show

$$c(k,t) = \frac{\widetilde{c}(k,t)}{|k|^m}, \quad u(x,t) = \sum_{k\neq 0} \frac{\widetilde{c}(k,t)}{|k|^{\gamma+m}} e^{ikx}. \tag{3.37}$$

for any positive integer m, where $\widetilde{c}$ may not be the same as in (3.36). Thus $u(\cdot,t) \in H^m(\mathbf{T})$. To show the regularity of u in t, we turn to (3.27), which implies that $c(k,t)$ is differentiable in t and

$$\frac{d}{dt} c(k,t) == (-\nu k^2 + i\alpha k^3)\, c(k,t) - i\, k|k|^\gamma \sum_{j\neq 0, j\neq k} \frac{c(j,t)c(k-j,t)}{|j|^\gamma\, |k-j|^\gamma}.$$

It then easily follows from (3.37) and Lemma 3.2 that

$$\left| \frac{d}{dt} c(k,t) \right| \le \frac{C}{|k|}.$$

This together with $u(\cdot,t) \in H^m(\mathbf{T})$ guarantee (3.35). This completes the proof of Theorem 3.5.

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