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Optimal Static Hedging of Currency Risk Using FX Forwards

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ABSTRACT: An exporter is invariably exposed to a currency risk due to unpredictable fluctuations in the exchange rates, and it is of paramount importance to minimize risk emanating from these forex exposures. In this paper, we present optimal static currency hedging strategies in which Forward FX contracts are used as hedging instruments. First, we introduce a static trading strategy and present the analytical expressions for different risk measures such as Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), Probability of Loss, and Conditional Expectation of Loss. The results presented here make no implicit assumptions about the underlying probability distribution. Next, using the expressions for risk measures we derive optimal static hedginging strategies to minimize these risk measures. Finally, we illustrate the results by specializing the underlying model to the case of geometric Brownian motion.

1 Introduction

Exposure to *foreign exchange (FX) risk* (Stephens, 2001; Xin, 2003) arises when companies conduct business in multiple currencies. A typical scenario such a company encounters may involve a future receivable in foreign currency (FC) for some service/goods the company exported. The FC thus received needs to be converted to home currency (HC) at some predetermined time. However, due to uncertain fluctuations in exchange rates, the company may incur huge losses (or reduced profits) at the time of conversion. Specifically, if the value of the HC appreciates against FC by the conversion date, then the company will receive less HC. On the other hand if FC appreciates then the company will profit from the exchange rate fluctuation. Since the direction and magnitude of these fluctuations are uncertain the exchange rate is justified to be classified as a risk. Furthermore, the magnitude of such fluctuations are significant enough to affect the company's profit and loss. The subject of analysis and mitigation of such financial risk factors comes under risk management (Alexander, 2009; Dowd, 2005; Jorion, 2006; Holton, 2003) and the specific case of exchange rate risk comes under currency risk management (Stephens, 2001; Xin, 2003).

Risk management consists primarily of an accurate analysis of the risk factors involved (FX rate, in this case) and take appropriate action to mitigate the risk. Any such action is known as *hedging*. There are many alternatives available to the company for hedging including conversion of FC to HC at an appropriate time or enter into different hedging contracts with some financial institution. Such contracts may involve FX instruments (also known as *hedges*) such as *forwards*, *futures*, *swaps*, and a variety of *FX options* (Stephens, 2001; Xin, 2003). Specifically, a *FX forward* is an agreement between the two parties to exchange currencies, namely, to buy or sell a particular currency at a predetermined future date and a predetermined exchange rate. It costs nothing to enter a forward contract. The party agreeing to buy the currency in the future assumes a *long position*, and the party agreeing to sell the currency in the future assumes a *short position*. The price agreed upon is called the delivery price, which is equal to the forward rate at the time of initiation of the contract. An *option* sets a rate at which the company may *choose* to exchange currencies. If the exchange rate at option maturity is more favorable, then the company will not exercise this option. Forward FX contracts are considered to be the simplest and the most widely used instrument for currency hedging. Their popularity may be partially explained by their simplicity, their initial zero cost, and feasibility of over-the-counter (OTC) trading that permits exact specifications regarding dates and amounts. For a general introduction to various (equity-based) financial instruments and their utility in hedging, see (Wilmott, 2007; Hull, 2008).

A *hedging strategy* represents a trading strategy involving different types of hedges that can reduce currency risk. The hedging strategy typically depends on information such as exchange rate fluctuations and possibly a predictive model for the exchange rate (the so called *market view*). These hedging schemes involve forward-rate based transactions to essentially lock in future exchange rate. Suppose a company is expecting a certain amount of FC at a future date. The company has a choice of doing nothing (*no hedging*), enter into a hedging contract once (*static hedging*), or enter into multiple contracts at different times (*dynamic hedging*). The choice depends on various reasons including the companies view on the market and service costs to enter hedging contracts. If the company can eliminate or achieve minimum possible risk using no hedging or static hedging then dynamic hedging offers no advantage. However, in highly volatile periods, dynamic hedging may achieve significantly less risk compared to static hedging. See (Wilmott, 2007; Hull, 2008) for a detailed introduction to hedging.

In summary, the first step in a risk management process to derive a predictive model for the FX rate. The next step is to identify an appropriate risk measure/metric (Artzner et al., 1999; Alexander, 2009; Dowd, 2005; Rockafellar and Uryasev, 2000), a number that can be used to quantify or summarize the effect of the risk from the companys exposure

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to FC. There are many risk measures available to quantify such risk exposure. In this paper, we consider six measures of risk including the expectation, the variance, the probability of loss, the value-at-risk, the condition expectation, and the conditional value-at-risk. Note, that these risk metrics try to capture the effect of the randomness of the risk factors on profit and loss and hence computation of risk metrics will involve probability-based methods. The final step in this process is to determine an optimal hedging strategy. An optimal hedging problem is a constrained optimization problem that minimizes a risk metric over a feasible set of decision variables, which are typically used to update the hedging portfolio.

Currency risk hedging continues to remain an active area of research (Stephens, 2001; Xin, 2003; Gagnon et al., 1998; Lypny and Powalla, 1998; Campbell et al., 2010; Glen and Jorion, 1993; Chang, 2011). Most of the existing literature focuses on specific risk measures such as variance or conditional value-at-risk (Topaloglou et al., 2002). However, (Volosov et al., 2005; Bhatia et al., 2012) focuses on variety of risk measures which relies on Monte Carlo simulations. In this paper, we present the problem of designing risk-optimal discrete-time hedging strategies for a given risk measure. Here, our primary focus is on optimal static hedging problem using Forward FX contracts as a hedging instrument although the overall methodology can easily be extended to include other instruments.

2 Notation

In this section, we introduce the notation used in this paper. Let $\mathbb{R}$ ($\mathbb{R}_+$) denote the set of (positive) real numbers. Next, let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, where Ω is a measurable space, $\mathbb{F}$ is a $\sigma-$algebra on Ω, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathbb{F})$. Let $L(\Omega, \mathbb{F}, \mathbb{P})$ be the set of integrable random variables on $(\Omega, \mathbb{F}, \mathbb{P})$. Given $X \in L(\Omega, \mathbb{F}, \mathbb{P})$, $\mathbb{E}[X]$ and $\mathrm{var}(X)$, denote the expectation (or the expected value) of X and the variance of X, respectively (under the measure $\mathbb{P}$). For an event $F \in \mathbb{F}$, we use $1_F(\omega)$ (or $1_{\omega\in F}(\omega)$) to denote the indicator function defined by

$$1_F(\omega) = \begin{cases} 1, & \omega \in F, \\ 0, & \text{otherwise.} \end{cases}$$

Next, for $F \in \mathbb{F}$, we use $\mathbb{E}[X|F]$ to denote the conditional expectation of X given the event F defined by

$$\mathbb{E}[X|F] = \frac{\mathbb{E}[X 1_F(\omega)]}{\mathbb{E}[1_F(\omega)]} = \frac{\mathbb{E}[X 1_F(\omega)]}{\mathbb{P}(F)}.$$

Note it is standard to define events in terms of inequalities involving some random variable X. For example, $F = \{\omega \in \mathbb{R} : X \geq 0\}$ denotes the event where $X \geq 0$. In this case, we denote F as $X \geq 0$ and the corresponding indicator function as $1_{X\geq 0}$ where the dependence on ω is implicit. In this case, we write $\mathbb{P}(X \geq 0)$, $1_{X\geq 0}$, and $\mathbb{E}[X|X \geq 0]$ to denote $\mathbb{P}(F)$, $1_F(\omega)$, and $\mathbb{E}[X|F]$, respectively. Furthermore, for a given continuous random variable X, we denote its probability density function by $f_X(x)$ and its probability distribution function by $F_X(x)$. Finally, we use $\Phi(x)$ and $\phi(x)$ to denote the standard normal distribution function and its corresponding density function, respectively.

3 The Loss Random Variable under a Static Hedging Strategy

Consider a company that is expecting a foreign currency of $N \in \mathbb{R}_+$ units at a future time $T \in \mathbb{R}_+$ (we denote the current time to be $T_0 (< T)$) where $T_0 \in \mathbb{R}$, $T_0 \geq 0$). Furthermore, we assume that the company has an obligation to pay NB units of HC at time T, where $B \in \mathbb{R}_+$ is the *budget rate*. Typically the budget rate B is determined by the company once a year based on many factors such as costs and profit expectations. Let us denote by X_t the exchange rate in number of HC units for one unit of FC at time $t \in \mathbb{R}_+$. If the company converts all the FC into the HC at time T then the loss ψ_T at time T is given by

$$\psi_T = N(B - X_T). \tag{1}$$

This represents the *no hedging strategy* where a positive ψ_T represents a loss and a negative ψ_T represents a profit. Note that if ψ_T is beyond the allowable limits then the company has to employ a hedging strategy to manage its risk exposures. In this paper, we consider only those hedging strategies that involve setting up a hedging portfolio of liquid Forward FX contracts.

Consider a hedging portfolio that consists of the positions in FX forward contracts only. We assume that all of these contracts are initiated at time T_0 with an expiration at a future time T. We further assume that we can make changes to this hedging portfolio only at the discrete times $0 = T_0 < T_1 < \cdots < T_n = T$ by including the new positions of these contracts within the hedging portfolio. Let $F_{k,T}, k \in \{0, \ldots, n-1\}$ denote the spot forward FX price (or forward rate) at time $T_k \in \mathbb{R}_+$ of a forward contract which is initiated at time T_0 and matures at T. Let $\zeta_k \in \mathbb{R}$ represent short position in FX forward contract bought at $T_k, k \in \{0, \ldots, n-1\}$. This arrangement allows the company to sell FC currency at an exchange rate of $F_{k,T}$ at time T. At the next time T_{k+1}, ζ_{k+1} units of this contract, with a forward rate of $F_{k+1,T}$, are bought. At T, all of the FX forward contracts held in the hedging portfolio mature and generate a combined payoff of $\sum_{i=0}^{n-1} \zeta_i F_{i,T}$ and any remaining $N - \sum_{i=0}^{n-1} \zeta_i$ units of FC are then spot traded at X_T. Under these trading rules the loss from this hedging portfolio will take the form

$$\begin{aligned} \psi_T &= NB - \left(\sum_{i=0}^{n-1} \zeta_i F_{i,T} + (N - \sum_{i=0}^{n-1} \zeta_i) X_T \right) \\ &= NB - \left(\sum_{i=0}^{n-1} \zeta_i (F_{i,T} - X_T) + N X_T \right). \end{aligned} \tag{2}$$

The expression in (2) represents a loss from discrete time *dynamic hedging strategy*. In this paper, we restrict our attention to a *static hedging strategy* where the company can invest only once, at time T_0, in the FX forward contracts and hold the portfolio till T. Under the static trading assumption, $\zeta_k = 0$, for $k \in \{1, \ldots, n-1\}$. Then the loss in (2) will specialize to

$$\begin{aligned} \psi_T &= NB - (\zeta(F_{0,T} - X_T) + NX_T) \\ &= NB - \zeta F_{0,T} - (N - \zeta)X_T. \end{aligned} \tag{3}$$

where $\zeta \triangleq \zeta_0$. In (3), the company is buying ζ units of the forward contract at forward rate $F_{0,T}$. This arrangement allows the company to sell FC currency at an exchange rate of $F_{0,T}$ at time T. By locking into a forward contract to sell FC, the company sets a future exchange rate without any additional cost. The remaining $(N - \zeta)$ units of FC is then exchanged at T (at spot rate). Using this strategy, the company now has a choice of i) covering all the N units of FC, that is, entering into $\zeta = N$ forward contracts each of unit value, ii) covering none of the FC (the *no hedging* or the (trivial) static hedging strategy with $\zeta = 0$), or iii) cover $\zeta \neq N$ units of FC. The hedging portfolio is said to be *under-hedged* if $\zeta < N$, *fully-hedged* if $\zeta = N$, and *over-hedged* if $\zeta > N$.
Since the loss ψ_T is a function of the random variable X_T it follows that ψ_T is also a random variable and the distribution of ψ_T is induced by X_T. Now, we assume that the underlying variable X_T is a continuous random variable with a known probability distribution function F_{X_T}. The following result relates the distribution and the density functions of the loss function in (3) with F_{X_T}.

Proposition 3.1. *Consider the loss random variable ψ_T given by (3) under a static hedging strategy defined by ζ. Then the probability distribution function of ψ_T,is given by*

$$\mathrm{F}_{\psi_T}(x) = \begin{cases} 1 - \mathrm{F}_{X_T}\left(\frac{NB - \zeta F_{0,T} - x}{N - \zeta}\right), & x \in \mathbb{R}, & \zeta < N, \\ \mathrm{F}_{X_T}\left(\frac{NB - \zeta F_{0,T} - x}{N - \zeta}\right), & x \in \mathbb{R}, & \zeta > N, \\ 1, & x \geq N(B - F_{0,T}), & \zeta = N \\ 0, & x < N(B - F_{0,T}), & \zeta = N. \end{cases} \tag{4}$$

Furthermore, if X_T has a probability density function then for $\zeta \neq N$, ψ_T has a probability density function given by

$$f_{\psi_T}(x) = \frac{1}{|N - \zeta|} f_{X_T}\left(\frac{NB - \zeta F_{0,T} - x}{N - \zeta}\right), \quad x \in \mathbb{R}, \quad \zeta \neq N. \tag{5}$$

Proof. Let $x \in \mathbb{R}$. Note that

$$\begin{aligned} \mathrm{F}_{\psi_T}(x) &= \mathbb{P}(\psi_T \leq x) \\ &= \mathbb{P}\left(NB - \zeta F_{0,T} - (N - \zeta)X_T \leq x\right) \\ &= \mathbb{P}\left((N - \zeta)X_T \geq NB - \zeta F_{0,T} - x\right) \end{aligned} \tag{6}$$

where the second equality follows from (3). If $\zeta < N$ then (6) yields

$$\begin{aligned} \mathrm{F}_{\psi_T}(x) &= \mathbb{P}\left(X_T \geq \frac{NB - \zeta F_{0,T} - x}{N - \zeta}\right) \\ &= 1 - \mathrm{F}_{X_T}\left(\frac{NB - \zeta F_{0,T} - x}{N - \zeta}\right) \end{aligned}$$

and if $\zeta > N$ then

$$\begin{aligned} \mathrm{F}_{\psi_T}(x) &= \mathbb{P}\left(X_T \leq \frac{NB - \zeta F_{0,T} - x}{N - \zeta}\right) \\ &= \mathrm{F}_{X_T}\left(\frac{NB - \zeta F_{0,T} - x}{N - \zeta}\right). \end{aligned}$$

Finally, if $\zeta = N$ then ψ_T is deterministic and has the value $N(B - F_{0,T})$. This completes the proof of (4).
It is clear from (4) that in the cases where $\zeta \neq N$, the cumulative distribution function of ψ_T is differentiable if F_{X_T} is differentiable. Differentiating (4) for case $\zeta \neq N$ yields (5). □

4 Expressions for Measures of Risk due to the Loss

In this section, we develop expressions for a variety of measures of risk due to loss. In this paper, we define a *risk measure* of a random variable as a real value function of the random variable to measure the risk of that random variable. Specifically, we consider six measures of risk including the expectation, the variance, the probability of loss, the value-at-risk, the condition expectation, and the conditional value-at-risk.
The following result provides expressions for the expectation and the variance of the loss (3).

Proposition 4.1. *Consider the loss random variable ψ_T given by (3) under a static hedging strategy defined by ζ. Then the expected loss $\mathbb{E}[\psi_T]$ is given by*

$$\mathbb{E}[\psi_T] \quad = \quad NB - \zeta F_{0,T} - (N-\zeta)\mathbb{E}[X_T] \tag{7}$$

and variance of the loss $\mathrm{var}(\psi_T)$ *is given by*

$$\mathrm{var}(\psi_T) \quad = \quad (N-\zeta)^2 \mathrm{var}(X_T). \tag{8}$$

Proof. (7) and (8) follow from (3) by applying standard rules of $\mathbb{E}[\cdot]$ and $\mathrm{var}(\cdot)$. □

Next, we consider the risk measure given by *probability of loss*. This measure assigns number denoting the probability that loss exceeds a prescribed level. The following result provides the expression for the probability of loss given by (3).

Proposition 4.2. *Consider the loss random variable ψ_T given by (3) under a static hedging strategy defined by ζ. Then, for $\theta \in \mathbb{R}$, the probability of loss $\mathbb{P}(\psi_T > \theta)$ is given by*

$$\mathbb{P}(\psi_T > \theta) = \begin{cases} \mathrm{F}_{X_T}\left(\frac{NB-\zeta F_{0,T}-\theta}{N-\zeta}\right), & \zeta < N \\ 1 - \mathrm{F}_{X_T}\left(\frac{NB-\zeta F_{0,T}-\theta}{N-\zeta}\right), & \zeta > N \\ 0, & \theta \geq N(B-F_{0,T}), \quad \zeta = N \\ 1, & \theta < N(B-F_{0,T}), \quad \zeta = N. \end{cases} \tag{9}$$

Proof. Note that $\mathbb{P}(\psi_T > \theta) = 1 - \mathrm{F}_{\psi_T}(\theta)$ so that (4) implies (9). □

Risk management has received much attention from practitioners, regulators and researchers in the last few years. *Value-at-Risk* (VaR) has emerged as a popular risk measurement tool among practitioners and regulators. In general, for a given confidence level $\alpha \in (0,1)$, VaR can be described as the maximum likely loss to occur in $(1-\alpha)\%$ of cases. For example, a bank might claim that its weekly VaR of the trading portfolio is \$50 million at the 99% confidence level. In other words, over the course of the next week, there is a 99 percent chance that the bank will not lose more than \$50 million. Alternatively, there is 1% chance i.e. only 1 chance in a 100, for a loss greater than \$50 million to occur.

Mathematically, for a given confidence level $\alpha \in (0,1)$ and a loss random variable Y, the VaR of the portfolio at the confidence level α is given by the smallest number y such that the probability that the loss Y exceeds y is at most $(1-\alpha)$. That is,

$$\mathrm{VaR}_\alpha(Y) \quad = \quad \inf\{y \in \mathbb{R} : \mathbb{P}(Y > y) \leq 1-\alpha\} \tag{10}$$

If we assume that the loss random variable Y has an associated probability distribution function F_Y. Then the VaR in (10) can be written as

$$\mathrm{VaR}_\alpha(Y) \quad = \quad \inf\{y \in \mathbb{R} : \mathrm{F}_Y(y) \geq \alpha\}$$

Under the assumption that $\mathrm{F}_Y(y)$ is continuous and strictly increasing, the infimum in (10) is attained and there exists y such that $\mathrm{F}_Y(y) = \alpha$.

Remark 4.1. $\mathrm{VaR}_\alpha(Y)$ *is also considered as the α-quantile of the* F_Y*, that is,*

$$\mathrm{VaR}_\alpha(Y) = \mathrm{F}_Y^{-1}(\alpha). \tag{11}$$

Remark 4.2. *The confidence level α in (10) is usually 95% or more. When $\alpha \in (50\%, 100\%)$, then it is easy to observe that* $\mathrm{VaR}_{1-\alpha}(Y) < \mathrm{VaR}_\alpha(Y)$.

The following result provides the expression for the VaR_α for the loss given in (4).

Proposition 4.3. *Consider the loss random variable ψ_T given by (3) under a static hedging strategy defined by ζ. Then, for $\alpha \in (0,1)$,* $\mathrm{VaR}_\alpha(\psi_T)$ *is given by*

$$\mathrm{VaR}_\alpha(\psi_T) = \begin{cases} NB - \zeta F_{0,T} - (N-\zeta)\mathrm{VaR}_{1-\alpha}(X_T), & \zeta \leq N \\ NB - \zeta F_{0,T} - (N-\zeta)\mathrm{VaR}_\alpha(X_T), & \zeta \geq N. \end{cases} \tag{12}$$

Proof. (12) follows from (4) by noting that $\mathrm{VaR}_\alpha(\psi_T) = \mathrm{F}_{\psi_T}^{-1}(\alpha)$. □

VaR though possess certain attractive properties, however, lacks certain features which undermines its applicability in many problems. For example, VaR_α is a number that defines a level of loss that one is reasonably sure will not be exceeded. However, it tells us nothing about the extent of the extreme losses that could be incurred in the event that VaR is exceeded. *Conditional Value-at-Risk* (CVaR) provides the information about the average level of loss, given that VaR is exceeded. Mathematically, for a given confidence level $\alpha \in (0,1)$ and a random variable Y, the CVaR is defined as

$$\mathrm{CVaR}_\alpha(Y) \quad = \quad \mathbb{E}[Z | Z > \mathrm{VaR}_\alpha(Y)]. \tag{13}$$

Since (13) is a conditional expectation, CVaR can be obtained by dividing the probability weighted average of the random variable Y that is greater than $\mathrm{VaR}_\alpha(Y)$ by $\mathbb{P}(Y > \mathrm{VaR}_\alpha(Y))$. But $\mathbb{P}(Y > \mathrm{VaR}_\alpha(Y)) = 1-\alpha$. Hence, if the loss random variable Y has a probability density function f_Y, then CVaR is given by

$$\mathrm{CVaR}_\alpha(Y) \quad = \quad \frac{1}{1-\alpha}\int_{\mathrm{VaR}_\alpha(Y)}^{\infty} y f_Y(y)\mathrm{d}y. \tag{14}$$

By definition, CVaR can never be less than the VaR. The difference between the CVaR and the corresponding VaR depends on the heaviness of the left tail of the loss distribution. The heavier this tail, the greater the difference. Clearly, CVaR measure gives a better description of the risks of a portfolio than just reporting the VaR.

We can also define CVaR by using the definition of conditional expected loss. The *conditional expected loss* can be defined as a mean loss which is above some prescribed loss level. Mathematically, for some level $\theta \in \mathbb{R}$ and a continuous loss random variable Y, the conditional expected loss is defined as $\mathbb{E}[Y|Y > \theta]$. Using (13), the CVaR can be defined in terms of $\mathbb{E}[Y|Y > \theta]$ by replacing θ with $\mathrm{VaR}_\alpha(Y)$. In the following result, we present an expression for the conditional expected loss for a loss variable ψ_T in (3). Subsequently, we will use this result to specialize to CVaR.

Proposition 4.4. *Consider the loss random variable ψ_T given by (3) under a static hedging strategy defined by ζ. Then, for $\theta \in \mathbb{R}$, the conditional expected loss $\mathbb{E}[\psi_T|\psi_T > \theta]$ is given by*

$$\mathbb{E}[\psi_T|\psi_T > \theta] = \begin{cases} \vartheta - \frac{N-\zeta}{\mathrm{F}_{X_T}(\kappa)}\mathbb{E}[X_T 1_{\{X_T \le \kappa\}}], & \zeta \le N \\ \vartheta - \frac{N-\zeta}{1-\mathrm{F}_{X_T}(\kappa)}\mathbb{E}[X_T 1_{\{X_T \ge \kappa\}}], & \zeta \ge N. \end{cases} \tag{15}$$

where $\vartheta = NB - \zeta F_{0,T}$ and $\kappa = \frac{\vartheta-\theta}{N-\zeta}$.

Proof. Note that the loss variable ψ_T in (3) is given by

$$\begin{aligned} \psi_T &= NB - \zeta F_{0,T} - (N-\zeta)X_T, \\ &= \vartheta - (N-\zeta)X_T. \end{aligned}$$

Now, note that if $\zeta = N$ then $\psi_T > \theta$ if and only if $\vartheta > \theta$. Furthermore, if $\zeta = N$ ($\zeta > N$) then $\psi_T > \theta$ if and only if $X_T < \kappa$ ($X_T > \kappa$). Hence, the conditional expectation $\mathbb{E}[\psi_T|\psi_T > \theta]$ is given by

$$\mathbb{E}[\psi_T|\psi_T > \theta] = \begin{cases} \vartheta, & \theta < \vartheta \\ 0, & \text{otherwise.} \end{cases},$$

if $\zeta = N$,

$$\begin{aligned} \mathbb{E}[\psi_T|\psi_T > \theta] &= \mathbb{E}[\vartheta - (N-\zeta)X_T | X_T < \kappa] \\ &= \vartheta - \frac{N-\zeta}{\mathrm{F}_{X_T}(\kappa)}\mathbb{E}[X_T 1_{\{X_T \le \kappa\}}], \end{aligned}$$

if $\zeta < N$, and

$$\mathbb{E}[\psi_T|\psi_T > \theta] = \vartheta - \frac{N-\zeta}{1-\mathrm{F}_{X_T}(\kappa)}\mathbb{E}[X_T 1_{\{X_T \ge \kappa\}}],$$

if $\zeta > N$, which completes the proof of (15). □

The following result specializes (15) to $\mathrm{CVaR}(\psi_T)$.

Corollary 4.1. *Consider the loss random variable ψ_T given by (3) under a static hedging strategy defined by ζ. Then, for $\alpha \in (0,1)$, $\mathrm{CVaR}_\alpha(\psi_T)$ of the loss is given by*

$$\mathrm{CVaR}_\alpha(\psi_T) = \begin{cases} NB - \zeta F_{0,T} - \frac{N-\zeta}{1-\alpha}\mathbb{E}[X_T 1_{\{X_T \le \mathrm{VaR}_{1-\alpha}(X_T)\}}], & \zeta \le N \\ NB - \zeta F_{0,T} - \frac{N-\zeta}{\alpha}\mathbb{E}[X_T 1_{\{X_T \ge \mathrm{VaR}_\alpha(X_T)\}}], & \zeta \ge N. \end{cases} \tag{16}$$

Proof. The result is a direct consequence of (15) by replacing θ with $\mathrm{VaR}_\alpha(\psi_T)$. □

5 Minimum Risk Static Hedging Strategies

The *minimum risk static hedging problem* is the determination of an optimal hedging strategy in terms of the decision variable ζ that minimizes a given risk measure due to the loss. Specifically, let $\rho(\psi_T)$ denote a risk measure due to the loss ψ_T where ψ_T is a function og ζ. For example, $\rho(\psi_T) = \mathbb{E}(\psi_T|\psi_T > 0)$ or $\rho(\psi_T) = \mathbb{P}(\psi_T|\psi_T > 0)$ etc. The minimum risk static hedging problem can now be given by

$$\min_{\zeta \in \Lambda} \rho(\psi_T). \tag{17}$$

where $\Lambda \subseteq \mathbb{R}$ denotes the set of feasible static hedging strategies. The feasible set Λ is chosen by the company based on its risk appetite, risk tolerance, and regulatory constraints. In this section, we use the expressions developed in Section 4 to provide minimum risk static hedging strategies for risk measures including expected loss, probability of loss, and value-at-risk.

First, let the risk measure be the expected loss given by (7) and note that the expected loss $\mathbb{E}[\psi_T]$ is a linear function of the decision variable ζ. However, $\mathbb{E}[\psi_T]$ is a decreasing function if $F_{0,T} > \mathbb{E}[X_T]$ and an increasing function if $F_{0,T} < \mathbb{E}[X_T]$. If $\Lambda = \mathbb{R}$ (that is, unconstrained) then $\rho(\psi_T)$ is unbounded in either case and optimal ζ limits to $\pm\infty$. While this is a mathematically valid solution it should be noted that the variance becomes ∞ as ζ becomes $\pm\infty$. Hence, an unconstrained problem with expected loss as the risk measure is an impractical problem to solve. As an alternative, the company may impose constraints on the decision variable ζ or put constraints on its variance (leading to a mean-variance optimization problem). Λ may be chosen to be a closed interval to stasify either the variance constraints or absolute constraints on ζ itself. A typical regulatory constraint on ζ may be $0 \leq \zeta \leq N$ with the interpretation that no short-selling and over hedging are allowed. In this case, if $F_{0,T} > \mathbb{E}[X_T]$ then then the optimal ζ is N and if $F_{0,T} > \mathbb{E}[X_T]$ then then the optimal ζ is 0. The special case in which $F_{0,T} = \mathbb{E}[X_T]$ any value of ζ is optimal.

Next, consider the case where the risk measure is the variance of loss $\mathrm{var}(\psi_T)$ given by (8). Since $\mathrm{var}(\psi_T)$ is a quadratic function of ζ it follows that $\zeta = N$ is a global minimum (in the case $\Lambda = \mathbb{R}$). This is in agreement with our earlier observation that if $\zeta = N$ then the loss variable ψ_T is a deterministic function and $\mathrm{var}(\psi_T) = 0$.

Now, let the risk measure be the probability of loss given by (9). If $N(B - F_{0,T}) + \theta > 0$ then the probability of loss is increasing for the case $\zeta < N$ and decreasing for the case $\zeta > N$. In this case, a minimum exists if the feasible set Λ is closed and bounded. Similarly, if $N(B - F_{0,T}) + \theta < 0$ then the probability is decreasing for $\zeta < N$ and increasing for $\zeta > N$. Hence, in this case the global minimum exists at $\zeta = N$.

Finally, we consider the value-at-risk VaR given by (12). Note that $\mathrm{VaR}_\alpha(\psi_T)$ is a piecewise linear function of the decision variable ζ and continuous at $\zeta = N$. $\mathrm{VaR}_\alpha(\psi_T)$ has $i)$ a positive slope if $F_{0,T} < \mathrm{VaR}_{1-\alpha}(X_T)$, $ii)$ a negative slope on the left of N and a positive slope on the right of N, for the case $F_{0,T} \in (\mathrm{VaR}_{1-\alpha}(X_T), \mathrm{VaR}_\alpha(X_T))$, and $iii)$ a negative slope if $F_{0,T} > \mathrm{VaR}_\alpha(X_T)$. In all these cases the minimum exists if Λ is closed and bounded.

6 Specialization to Geometric Brownian Motion Model

In Section 3, we presented the expressions for the cdf and pdf associated with the P&L for a static hedging strategy and in Section 4 we use these functions to develop the expression for different risk measures. The results presented in these sections make no assumption of probability distribution for the underlying variable X_T. In this section, we assume that the underlying random variable, the exchange rate X_t at time t, evolves continuously in time and follows a log-normal distribution. Specifically, we assume that the exchange rate X_t evolves continuously in time according to the Itô stochastic differential equation

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma \mathrm{d}W_t \tag{18}$$

in the interval $[0, T]$, where $\mu > 0$ represents the rate of return, and $\sigma > 0$ is the volatility, both assumed to be constant and W_t is a Brownian motion. For a given initial value X_0, (18) has the solution given by

$$X_T = X_0 e^{\left((\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\epsilon\right)} \tag{19}$$

where ϵ is standard normal random variable corresponding to the Brownian motion W_T. Hence, it follows from (19) that X_T is a log-normal random variable given by

$$X_T \sim LN\left[\ln(X_0) + \left(\mu - \frac{1}{2}\sigma^2\right)T, \sigma\sqrt{T}\right] \tag{20}$$

and the expected value and the variance are respectively given by

$$\mathbb{E}(X_T) = X_0 e^{\mu T}, \tag{21}$$

$$\mathrm{var}(X_T) = X_0^2 e^{2\mu T}\left(e^{\sigma^2 T} - 1\right). \tag{22}$$

Next, the cumulative probability distribution of X_T is given by

$$\mathrm{F}_{X_T}(x) = \begin{cases} \Phi\left(\frac{\ln(x) - \ln(X_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right), & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{23}$$

with the corresponding probability density function given by

$$f_{X_T}(x) = \begin{cases} \frac{1}{\sqrt{2\pi}}\frac{1}{x\sigma\sqrt{T}}\exp\left(-\frac{\left(\ln(x) - \ln(X_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}\right), & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{24}$$

Now consider the static hedging problem having the P&L as defined in (3) and the associated probability distribution and the density functions as defined in (4) and (5) respectively. Under the assumption that the underlying random variable X_T follows a log-normal distribution, one can get the expressions for the distribution and the density functions of the loss variable ψ_T by substituting the log-normal cdf (23) and the log-normal pdf (24) in (4) and (5) respectively.

Similarly, the expressions for the expectation and the variance of loss variable ψ_T under the log-normal distribution assumption can be obtained by substituting (21) into (7) and (22) into (8) respectively, that is,

$$\begin{aligned} \mathbb{E}[\psi_T] &= NB - \zeta F_{0,T} - (N - \zeta)X_0 e^{\mu T} \\ \mathrm{var}(\psi_T) &= (N - \zeta)^2 X_0^2 e^{2\mu T}\left(e^{\sigma^2 T} - 1\right). \end{aligned}$$

Finally, the expression for $\mathrm{VaR}_\alpha(X_T)$ under the log-normal distribution assumption can be obtained by substituting the cdf of X_T (23) in (10), that is,

$$\mathrm{VaR}_\alpha(X_T) = X_0 \exp\left(\sigma\sqrt{T}\Phi^{-1}(\alpha) + (\mu - \frac{1}{2}\sigma^2)T\right) \tag{25}$$

By substituting (25) into (12) gives expression for the $\mathrm{VaR}_\alpha(\psi_T)$. Figure 1 illustrates the variation of $\mathrm{VaR}_\alpha(\psi_T)$ with ζ for different conditions involving $F_{0,T}$.

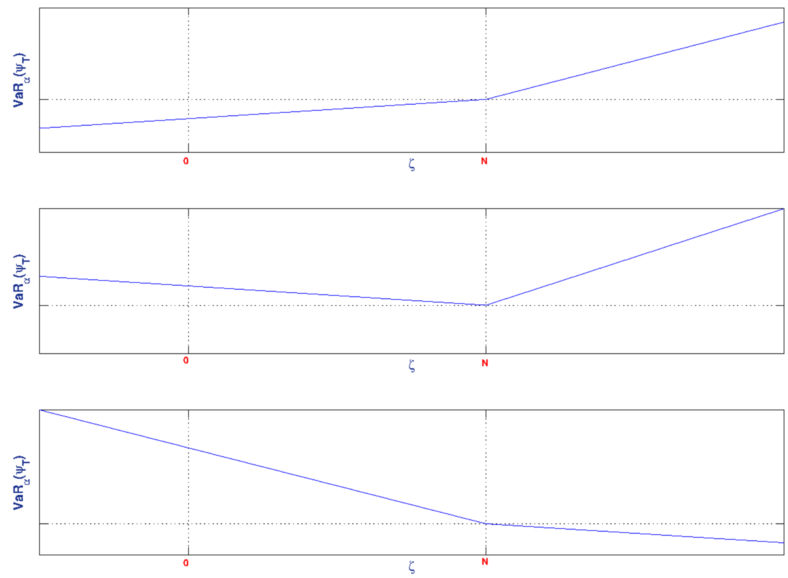

Figure 1: $\mathrm{VaR}_\alpha(\psi_T)$ vs ζ for $i)$ $F_{0,T} < \mathrm{VaR}_{1-\alpha}(X_T)$, $ii)$ $F_{0,T} \in (\mathrm{VaR}_{1-\alpha}(X_T), \mathrm{VaR}_\alpha(X_T))$, and $iii)$ $F_{0,T} > \mathrm{VaR}_\alpha(X_T)$

7 Conclusion

In this paper, we presented optimal static currency hedging strategies in which Forward FX contracts are used as hedging instruments. First, we introduced a static trading strategy and derived analytical expressions for different risk measures such as Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), Probability of Loss, and Conditional Expectation of Loss. The results presented here make no assumptions about the underlying probability distribution. Next, using the expressions for risk measures we derived optimal static hedginging strategies to minimize these risk measures.

It should be noted that in most cases, the optimal strategies were either the no-hedging or the fully-hedged cases. If the company is interested in a multi-objective minimization then the solution may be somewhere in between. The case of multi-risk minimization will be considered in a future paper. Furthermore, the current work will also be extended to the dynamic hedging case where the number of hedging times is sufficiently small for deriving expressions for risk measures and subsequently optimal hedging strategies.

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